Math 230 **Fall 2018**
Homework 3 **Drew Armstrong**

Fun With Axioms. In the following problems you are allowed to use the axioms from the handout and any results that we proved in class. You should document every step of your proofs, but it's okay to skip really boring things like commutativity and associativity.

Problem 1. In this problem you will prove that $(-a)(-b) = ab$ for all integers $a, b \in \mathbb{Z}$.

(a) Recall that $-a$ is the **unique** integer satisfying $a + (-a) = 0$. Prove that $-(-a) = a$.
(b) For all $a, b \in \mathbb{Z}$ prove that $(-a)b = a(-b) = -(ab)$. [Hint: Use the distribution axiom.]
(c) Recall that for all $b, c \in \mathbb{Z}$ we define $b - c := b + (-c)$. For all $a, b, c \in \mathbb{Z}$ prove that $a(b - c) = ab - ac$. [Hint: Use distribution and part (b).]
(d) For all $a, b \in \mathbb{Z}$ prove that $(-a)(-b) = ab$. [Hint: Use parts (a) and (b).]

Problem 2. Given an integer $a \in \mathbb{Z}$ we define its *absolute value* as follows:

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

(a) Use this definition to prove that $|ab| = |a|\,|b|$ for all $a, b \in \mathbb{Z}$. [Hint: Problem 1.]
(b) For all $a \in \mathbb{Z}$ prove that $a \neq 0$ if and only if $|a| \geq 1$.

Problem 3. In class we saw that $\mathbb{Z}$ satisfies "additive cancellation." In this problem you will show that $\mathbb{Z}$ also satisfies a form of "multiplicative cancellation." (The proof is quite a bit harder because we are not allowed to divide.) To begin, let $n \in \mathbb{Z}$ and define the statement

$$P(n) := \text{" for all integers } m \geq 1 \text{ we have } mn \geq 1. \text{ "}$$

(a) Prove that $P(1)$ is true.
(b) For any positive integer $n \geq 1$ prove that $P(n) \Rightarrow P(n+1)$. Then it follows by induction that $P(n)$ is true for all $n \geq 1$. In other words, you have shown that

$$(m \geq 1 \wedge n \geq 1) \Rightarrow (mn \geq 1) \quad \text{for all } m, n \in \mathbb{Z}.$$

(c) Combine this with Problem 2 to prove for all $a, b \in \mathbb{Z}$ that

$$(a \neq 0 \wedge b \neq 0) \Rightarrow (ab \neq 0).$$

(d) Finally, prove for all integers $a, b, c \in \mathbb{Z}$ that

$$(ab = ac \wedge a \neq 0) \Rightarrow (b = c).$$

[Hint: Use 1(c) and the contrapositive of 3(c).]