Mon Aug 26

Memorial 205
MTH 230 E (MWF 12:20 – 1:10)
"Intro to Abstract Mathematics"
(A.K.A. Mathematics)

Drew Armstrong
Email: armstrong@math.miami.edu
Web: www.math.miami.edu/~armstrong
Office: Ungar 533

Office Hours: TBA.

Evaluation:

There will be ≈ 6 HW assignments
3 in-class exams
NO FINAL EXAM.

	Homework	25%
In Class	Exam 1	25%
	Exam 2	25%
	Exam 3	25%
		100%

This class is about Math.

Q: What is math?

My (provisional) Answer:

- math allows humans to <u>agree</u> on things
- it's a language
  but not a "natural language".
- consciously invented to be

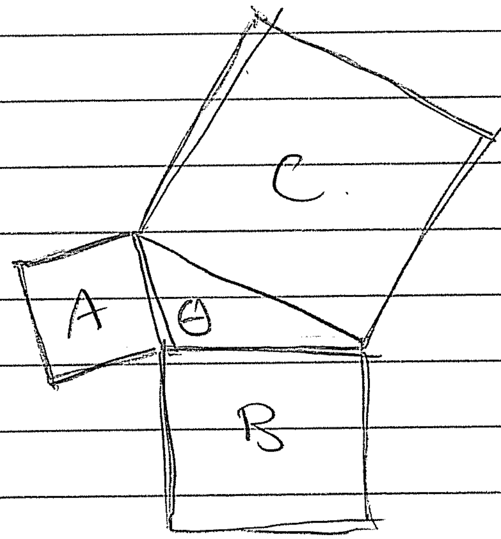
CLEAR & PRECISE.

"Rigor = Clarity + Precision"

- it uses English words but this is deceptive.

- you must learn math like you would a language.
  - immersion
  - practice
  - <u>open mind</u>.

Example:

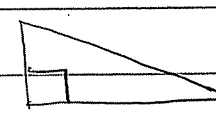


Claim: <u>If</u> $\theta = 90°$
<u>Then</u> area(C) = area(A) + area(B).

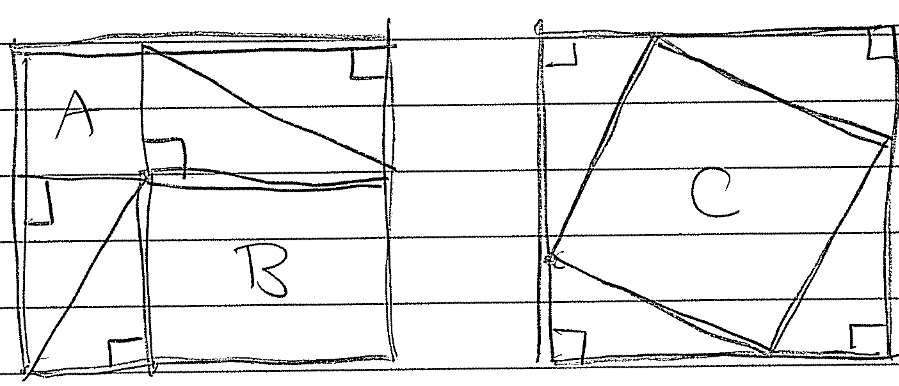
Why is this "true"?
<u>Is</u> it "true"?

I'll try to convince you.

<u>Proof</u>: <u>Assume</u> that $\theta = 90°$.



Observe the following two squares:

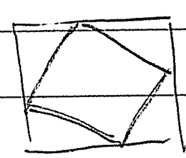


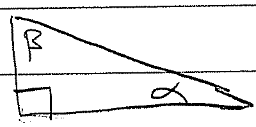
They have equal area.
Each contains 4 copies of the original triangle
If we remove the triangles,
what remains on each side must
still have equal area.
Hence

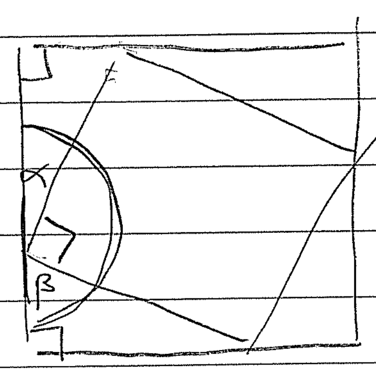
$$\text{area}(A) + \text{area}(B) = \text{area}(C)$$

Are you convinced?

Possible complaints:

- Why is  a square?

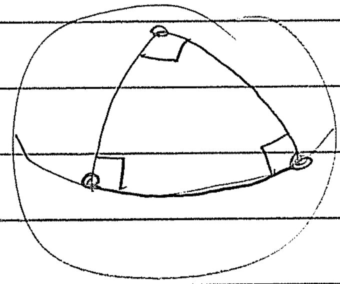




Because $\alpha + \beta + 90^\circ = 180^\circ$
(Angles in a triangle sum to 180°).

- Why is *that* true?

It's not true on a sphere (e.g. Earth).



three right angles!

$90° + 90° + 90° \neq 180°$.

OOPS!

Oh well. At least I showed that

(*if* $\theta = 90°$ *then* area(C) = area(A) + area(B))

$\Uparrow$

angles in a triangle sum to 180°

$\Uparrow$

Maybe you have more complaints.
- what is "area"? etc.

⋮

When can I stop?!

At some point I will "just stop"

Pythagorean Theorem

$\Uparrow$

Angles in a triangle sum to 180°

$\Uparrow$

$\vdots$

$\Uparrow$

$\boxed{\text{AXIOMS}}$

Hopefully we agree on some axioms
– they should be "self-evident"
(need no proof)

If you <u>still</u> don't agree, that's "your problem".

The axiomatic/deductive method arose in Greece ~ 600 BC
(Thales of Miletus, 625 BC — 546 BC)
and reached its full expression ~ 300 BC
(Euclid of Alexandria, ? – ?).

Wed Aug 28

MTH 230 ("Rigor = Clarity + Precision")

Drew Armstrong

www.math.miami.edu/~armstrong

Q: What is Math?

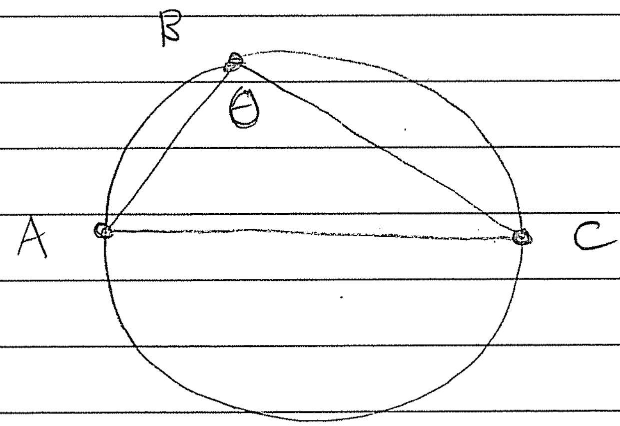
My Answer:

- math allows humans to agree on things
- a PROOF is an attempt to persuade
- a THEOREM is a thing we're been persuaded to agree about.

Example: The oldest Theorem

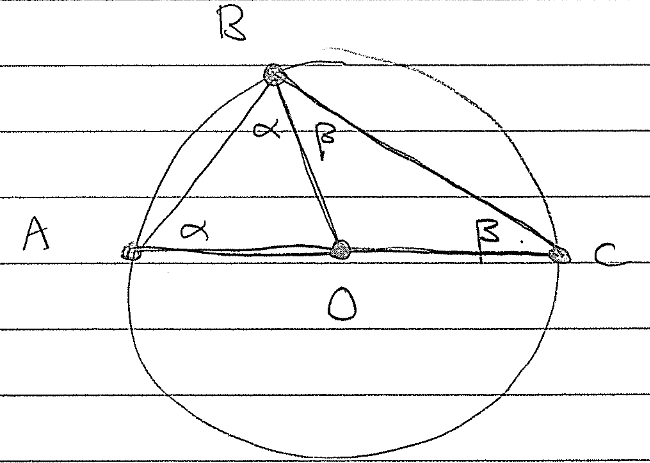
Thales' Theorem (~600 BC):

Consider points A, B, C on a circle



If AC is a diameter, then $\theta = 90°$.

Proof: Assume that AC is a diameter and draw the center O.



Draw the line segment OB.
Note that $\overline{OA} = \overline{OB} = \overline{OC}$ by the definition of "circle". Hence $\triangle OAB$ and $\triangle OBC$ are isoceles triangles.

We conclude that $\measuredangle OAB = \measuredangle OBA = \alpha$
and $\measuredangle OCB = \measuredangle OBC = \beta$

Since the angles in $\triangle ABC$ sum to 180° we have

$$\alpha + (\alpha + \beta) + \beta = 180^\circ$$
$$2\alpha + 2\beta = 180^\circ$$
$$2(\alpha + \beta) = 180^\circ$$
$$\alpha + \beta = 90^\circ$$

$\blacksquare$

Are you persuaded?

There is still room for complaint.
For example: Why do the angles in ΔABC sum to $180°$?

How skeptical should we be?
What will we allow ourselves to assume without proof ("AXIOMS")

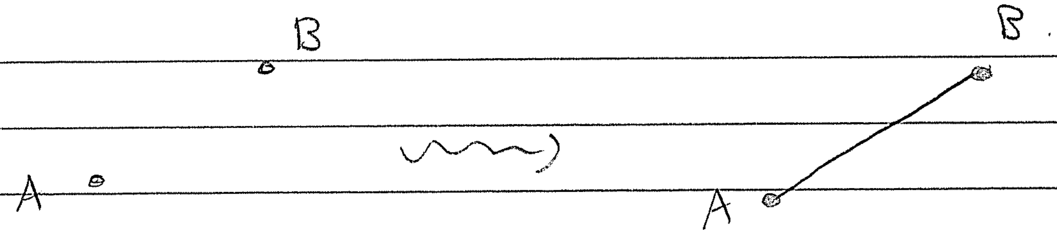
We should explicitly state our Axioms.
The first explicitly axiomatic system was given by Euclid ~300 BC

"The Elements"
- XIII Books
(Book I is a proof of the Pythagorean Theorem)
- Read by every educated person in the West until ~1900 (including Abraham Lincoln).
- Deduced (Proved) all of classical Greek mathematics from just

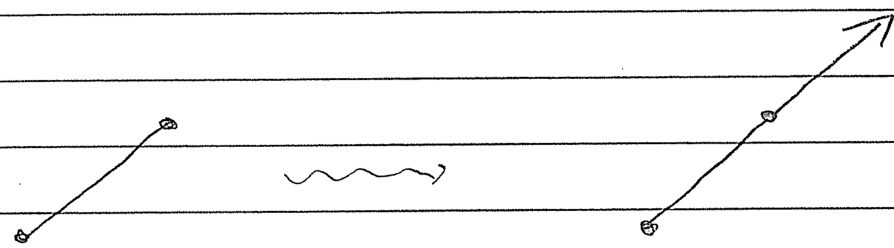
10 Axioms

10 Axioms $\left\{ \begin{array}{l} 5 \text{ "postulates"} \\ 5 \text{ "common notions"} \end{array} \right.$

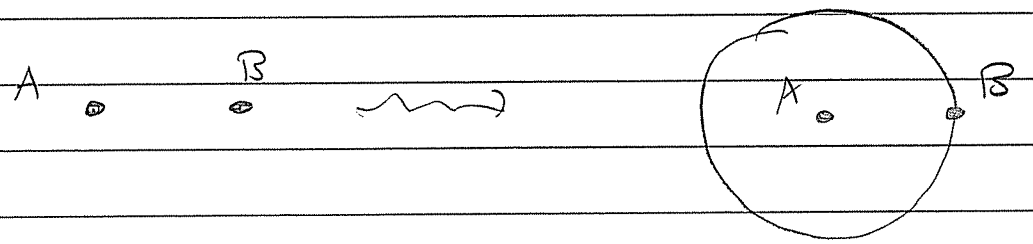
(P1)



(P2)

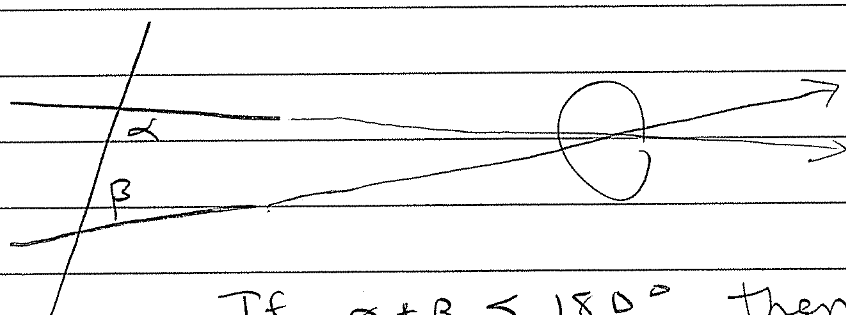


(P3)



(P4) All right angles are equal
(? Necessary ?).

(P5)



If $\alpha + \beta < 180°$ then the lines will meet on that side.

The common notions CN1 - CN5
describe properties of comparison

"=" and "<"

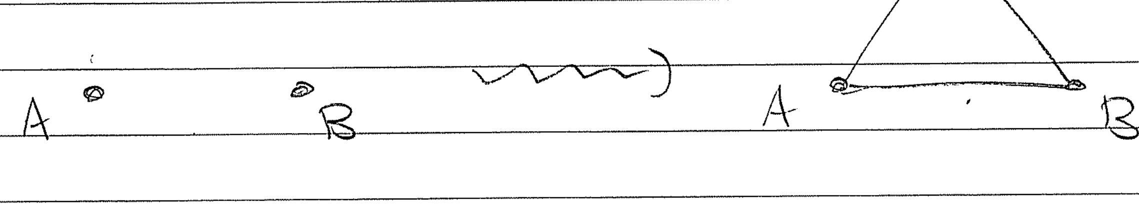
And that's all.
With a lot of work he deduced everything
from these axioms

Pythagorean Theorem
is Prop 47 in Book I.

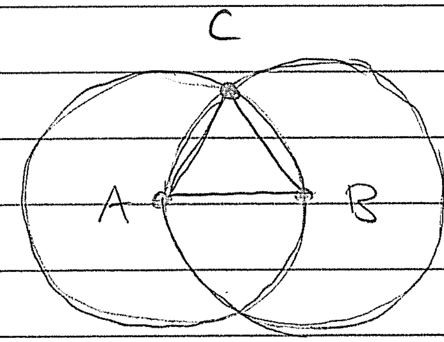
Thales' Theorem
is Prop 33 in Book III.

What is Prop 1 in Book I ?

Prop I.1: Given two points A, B
we can draw a point C such that
$\triangle ABC$ is equilateral



Proof:



Draw circle center A radius AB (P3)

Draw circle center B radius AB (P3)

Let C be a point of intersection of the two circles (?)

Draw triangle ABC.

We have $\overline{AC} = \overline{AB}$, $\overline{BC} = \overline{AB}$ } definition of "circle".

Hence also $\overline{AC} = \overline{BC}$ (CN 1)

$\therefore$ Triangle ABC is equilateral.

Q.E.D.

OOPS! Why does the point C exist?

Euclid gave no reason

(He was not perfect).

David Hilbert "fixed" The Elements in 1899. He needed 20 Axioms !

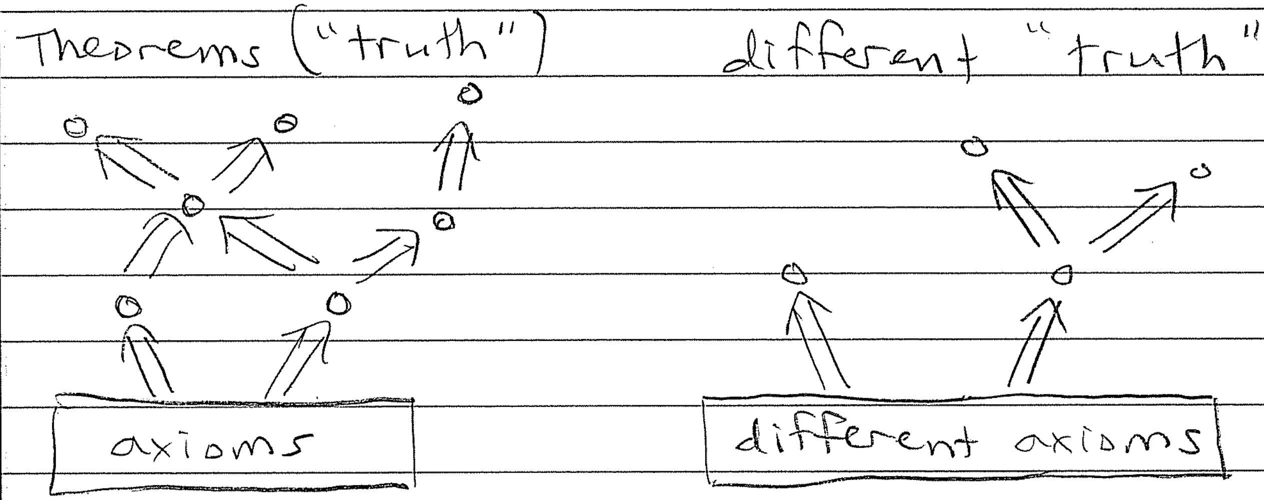
Fri Aug 30

The course webpage is up:

www.math.miami.edu/~armstrong/230fa13.html.

I will assign HW1 next week.
Assignment for the weekend:
Browse Euclid's Elements online.

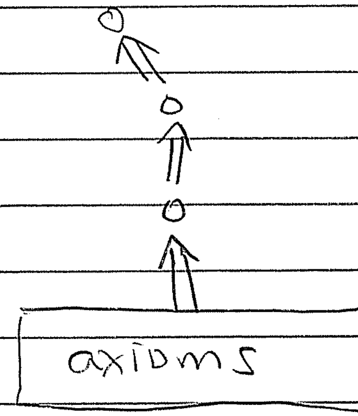
Schematic Diagram of Mathematics



- Theorems are statements that can be deduced logically from the axioms
- Different axioms, different notion of "truth"

A proof looks like this:

A sequence of logical deductions leading back to the axioms.



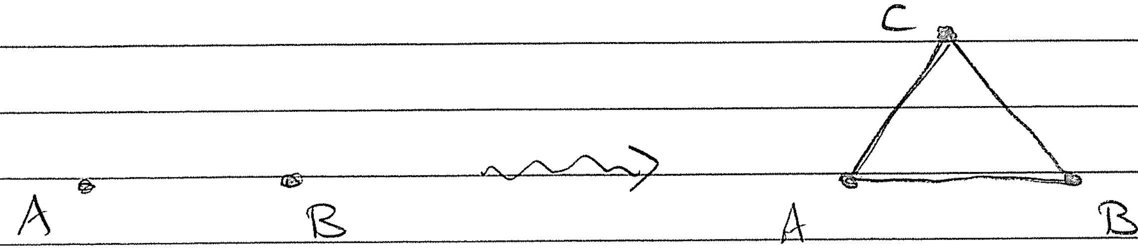
The first axiomatic system was Euclid's "Elements" (~300BC).

Axioms $\left\{ \begin{array}{l} 23 \text{ Definitions} \\ 5 \text{ Postulates} \\ 5 \text{ Common Notions} \end{array} \right.$

XIII Books containing 465 Propositions (i.e. Theorems).

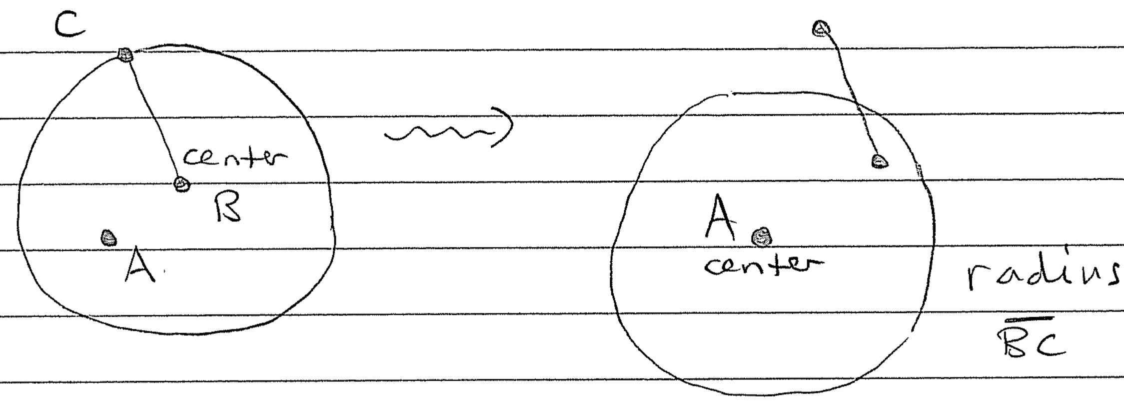
Recall

Prop I.1: To construct equilateral triangle

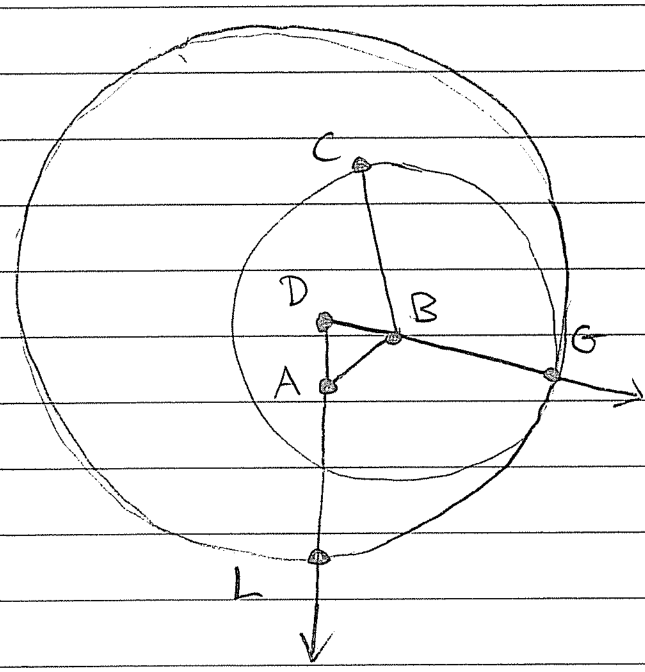


Prop I.2: To <u>move</u> a circle.

Given a circle center B radius BC, and a point A, we can draw a circle with center A and radius of length $\overline{BC}$.



Euclid's Proof:



Draw equilateral $\triangle ABD$ (I.1)

Extend DB to G (P2)

Draw circle center D radius DG (P3)

Extend DA to L (P2)

$\overline{DL} = \overline{DG}$ and $\overline{DA} = \overline{DB}$ Definitions of "circle" and "equilateral $\triangle$"

Hence

$$\overline{AL} = \overline{DL} - \overline{DA} = \overline{DG} - \overline{DB} = \overline{BG}$$ (CN3)

But $\overline{BC} = \overline{BG}$ Definition of "circle".

Hence $\overline{AL} = \overline{BG} = \overline{BC}$ (CN 1)

Finally,

Draw circle center A radius AL (P3)

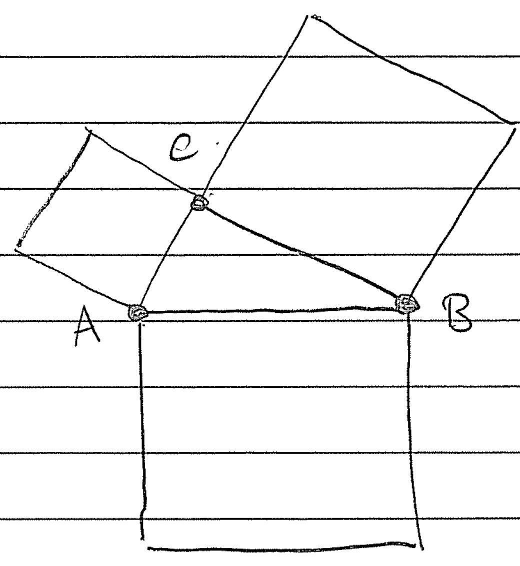
Q.E.D

Q: Why didn't Euclid just assume that you can pick up the compass and move it?

A: Because he didn't need to!

Book I contains 48 propositions

Prop I.47 is the Pythagorean Theorem.



If $\measuredangle ACB = 90°$

Then $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

///

So what is Prop I.48 ??

Prop I.48 is the CONVERSE of the Pythagorean Theorem:

If $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$ Then $\measuredangle ACB = 90°$

[We say $P \Rightarrow Q$ = "P implies Q"
= "if P then Q".

Note that

$P \Rightarrow Q$ and $Q \Rightarrow P$

are NOT logically equivalent.

Let P = "$x > 0$"
Q = "$x^2 > 0$"

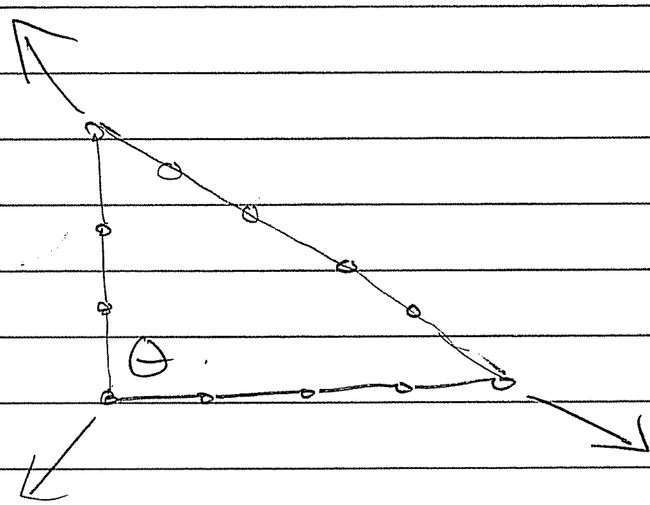
Then $P \Rightarrow Q$ is true
but $Q \Rightarrow P$ is False!

Counterexample: $x = -2$.
Then $x^2 > 0$ but $x \not> 0$.]

The converse of Pythag. Thm. has applications.

- Make a loop of rope with 12 equally spaced knots.

- Get two friends and pull.



- Since $3^2 + 4^2 = 9 + 16$
$$= 25 = 5^2$$

it follows that $\theta = 90°$.

- Build a pyramid!

Wed Sept 4

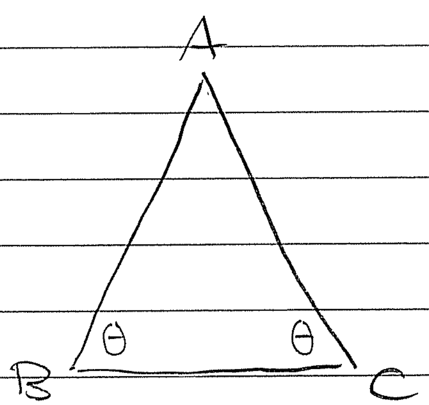
HW 1 due Fri Sept 13
at beginning of class

Office Hours (Ungar 533)

Mon 2–3 pm
Wed 3–4 pm
and by appointment.

Topic: Book I of Euclid.

Prop I.5 ("pons asinorum"):

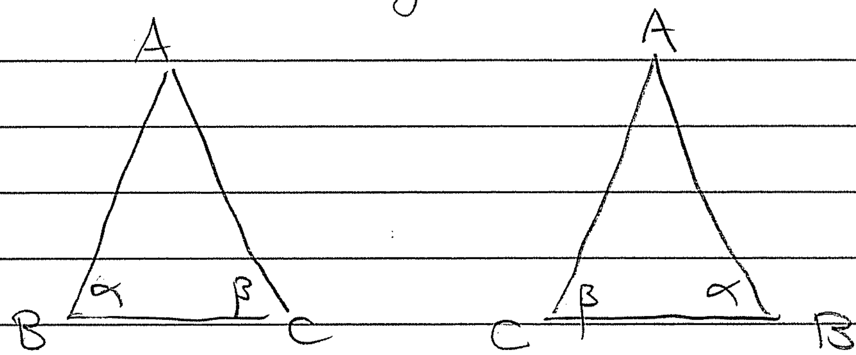


Consider $\triangle ABC$.
If $\overline{AB} = \overline{AC}$ (i.e. triangle is "isoceles")
then $\measuredangle ABC = \measuredangle BAC$

On HW 1 you will tell me Euclid's proof.

Here is a cute proof due to Pappus ($\sim$ 320 AD).

<u>Proof</u>: Consider the triangle and its mirror image



Since $\overline{AB} = \overline{AC}$,

$\overline{AC} = \overline{AB}$, and

$\angle BAC = \angle CBA$,

the triangles are congruent by the side-angle-side criterion (Prop I.4).

Hence $\alpha = \angle ABC = \angle ACB = \beta$.

$\blacksquare$

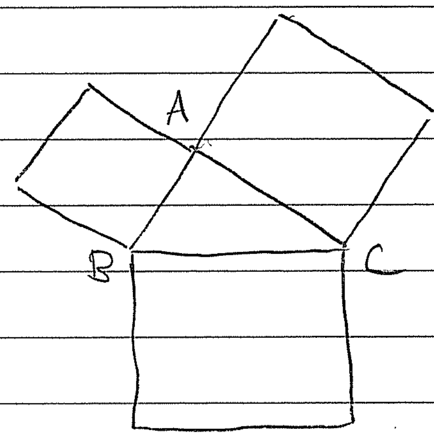
Prop I.32 states that:

the interior angles of any triangle sum to $180°$.

On HW 1, you will give an abbreviated proof. (I've given you the necessary ingredients.)

The final two propositions are

Prop I.47 (Pythagorean Theorem)



If $\measuredangle BAC = 90°$
then $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$

Proof omitted.

and

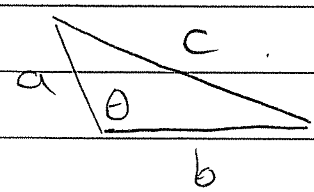
Prop I.48 (<u>converse</u> of Pyth. Thm.)

In the same diagram we have that

if $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$
then $\measuredangle BAC = 90°$

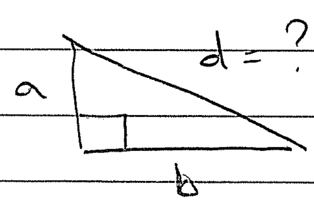
<u>Euclid's Proof in modern language</u> :

Label the sides and angle of the given Δ.



We <u>assume</u> that $a^2 + b^2 = c^2$ and we <u>want to show</u> that $\theta = 90°$.

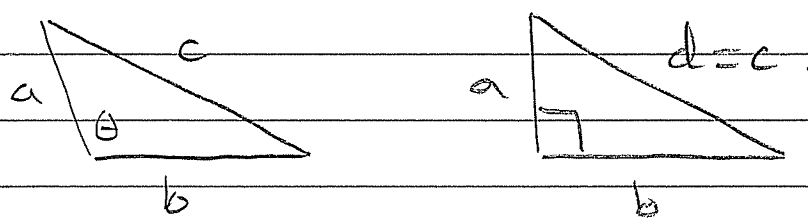
To do this we draw a (new) right triangle with side lengths a, b, d, where d is unknown to us



(Euclid I.2 and I.11 allow this.).

Then Prop I.47 (Pyth. Thm.) implies that $a^2+b^2=d^2$. But we *assumed* that $a^2+b^2=c^2$. Hence

$$c^2=d^2 \quad \text{and} \quad c=d.$$



The two triangles have the same side lengths, hence by Euclid I.8 ("side-side-side criterion for congruence") they have the same angles.

We conclude that $\theta = 90^\circ$ $\blacksquare$

Logical notation:

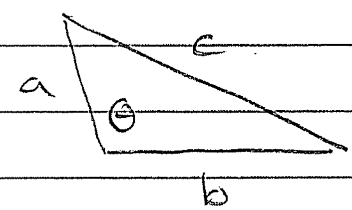
$$P \Rightarrow Q \;=\; \text{"}P \text{ implies } Q\text{"}$$
$$=\; \text{"if } P \text{ then } Q\text{"}$$
$$=\; \text{"}P \text{ only if } Q\text{"}$$

$P \Leftarrow Q$ = "P is implied by Q"
= "P if Q"

Then we say

$P \Longleftrightarrow Q$ = $P \Rightarrow Q$ AND $P \Leftarrow Q$.
= "P <u>if and only if</u> Q"

Putting Prop I.47 and I.48 together:



For any triangle we have

$$\theta = 90^\circ \Longleftrightarrow a^2 + b^2 = c^2$$

//

Fri Sept 6

HW 1 due next Friday

Office Hours : Mon 2-3 pm
Wed 3-4 pm

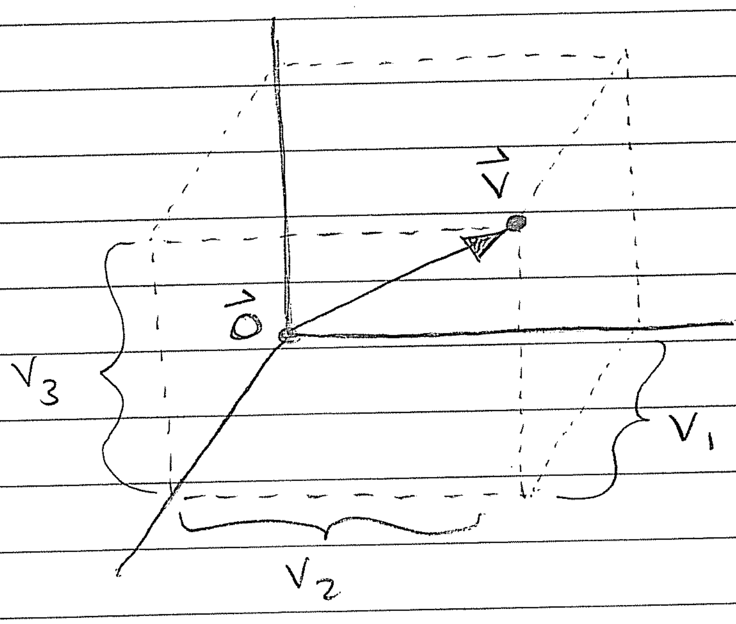
Today: Moving On

- The first axiomatic system was Euclid's Elements (~300 BC)

- The first modern mathematician was René Descartes (1596 - 1650).

Descartes' revolutionary idea :

a "point" $\equiv$ an ordered list of numbers



"a fly in the corner"

- fix 3 perpendicular axes
- given a point $\vec{v}$, imagine a rectangular box with opposite corners $\vec{0}$ and $\vec{v}$.
- if the dimensions of the box are v_1, v_2, v_3 we say.

$$\vec{v} = (v_1, v_2, v_3)$$

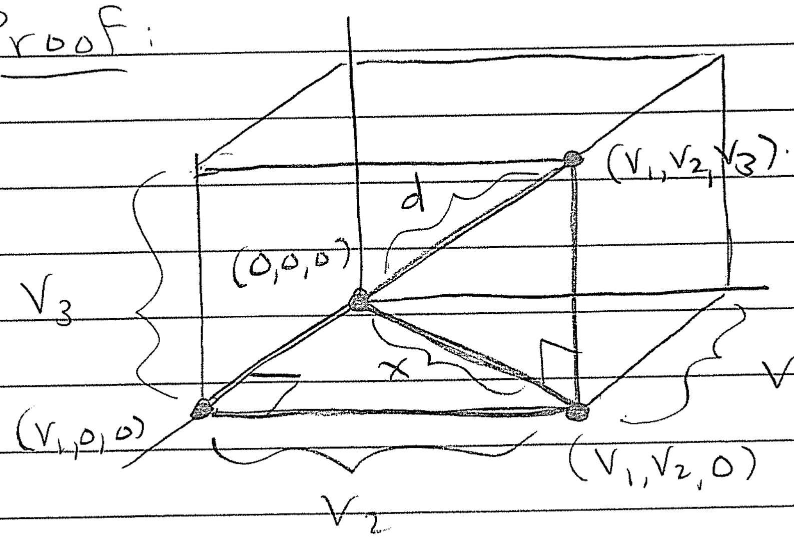
the "Cartesian coordinates" of the point.

- we also think of $\vec{v}$ as an arrow ("vector") with tail at $\vec{0} = (0,0,0)$.

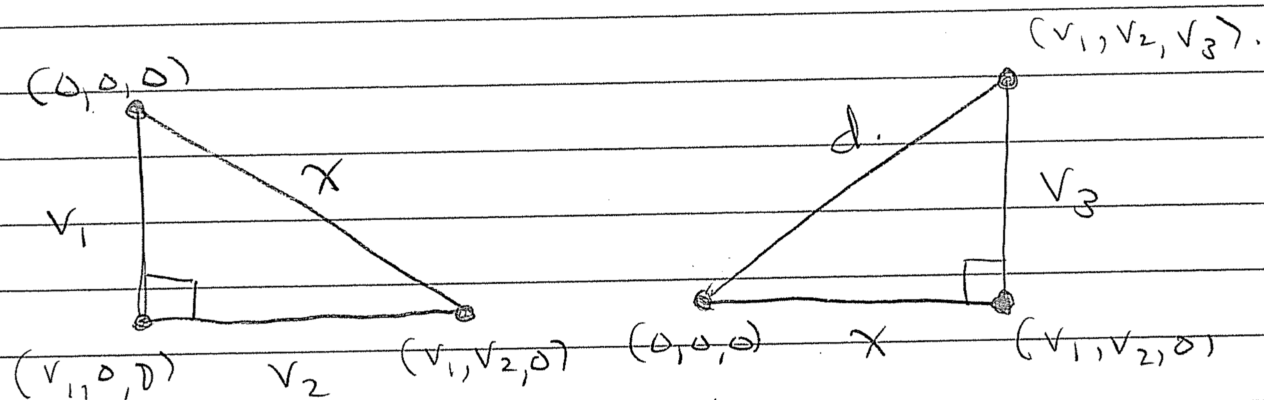
Claim: The length of the vector is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Proof:



Let $\|\vec{v}\| = d$ and consider the points $(0,0,0)$, $(v_1, 0, 0)$, $(v_1, v_2, 0)$, and (v_1, v_2, v_3).
We have two right triangles.



By Pythagoras we have

$$x^2 = v_1^2 + v_2^2 \quad \text{and} \quad d^2 = x^2 + v_3^2.$$

Combining the two equations gives

$$d^2 = x^2 + v_1^2 = (v_1^2 + v_2^2) + v^2$$

$$d^2 = v_1^2 + v_2^2 + v_3^2.$$

$$d = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Thinking: What is the length of a vector in 4D space?

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \quad ?$$

Here's something strange:

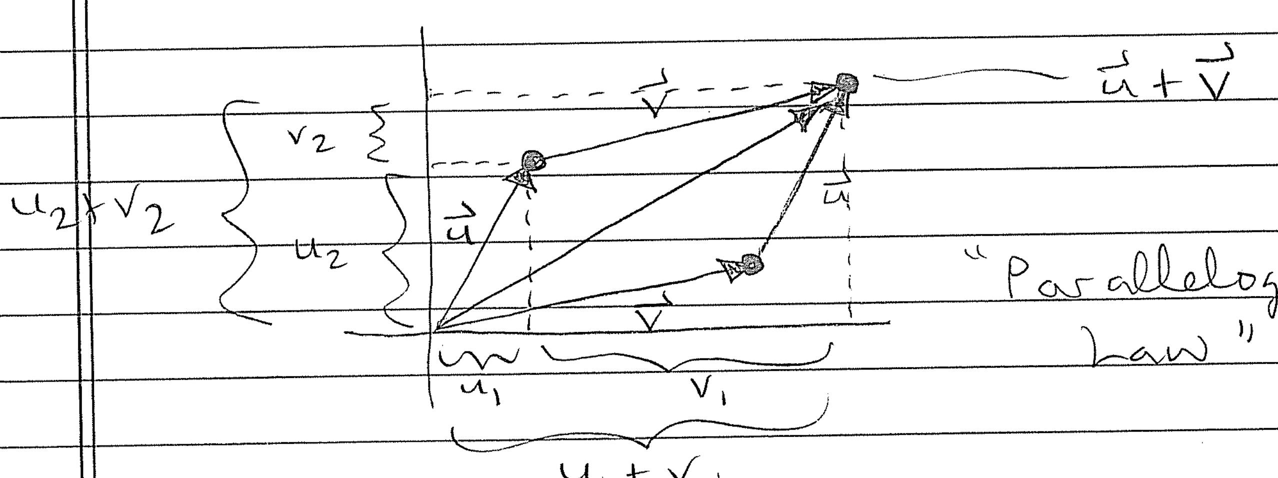
Since points are made of numbers, we can do algebraic things like "add" them:

$$\vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3)$$

$$:= (u_1 + v_1, u_2 + v_2, u_3 + v_3).$$

[What would Euclid think?]

Picture in 2D: $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$.



"Parallelogram Law"

Vectors add head-to-tail.

Q: What about subtraction?
Given $\vec{u}$ and $\vec{v}$, define

$$\vec{u} - \vec{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3).$$

What does it mean?
Two possibilities:

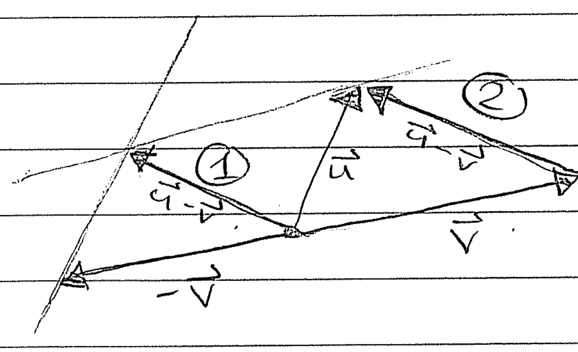
same length, opposite direction as $\vec{v}$.

① $\vec{u} - \vec{v} = \vec{u} +$ "$-\vec{v}$"

② $\vec{u} - \vec{v}$ is the vector $\vec{x}$ that satisfies

$$\vec{v} + \vec{x} = \vec{u}.$$

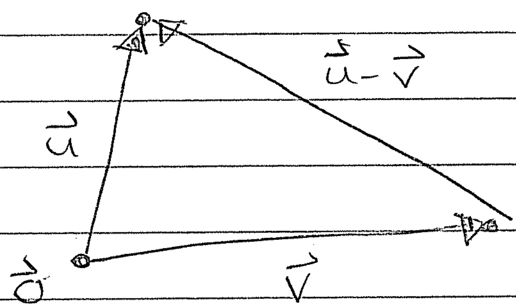
Picture:



① & ② are the SAME vector!

(A vector has direction and length, but not position.)

For any points $\vec{u}, \vec{v}$ we have a triangle:



We can compute the distance between $\vec{u}, \vec{v}$:

$$\operatorname{dist}(\vec{u}, \vec{v}) = \|\vec{u}-\vec{v}\|$$
$$= \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2}$$

Mon Sept 9

HW 1 due Friday

OH: Today 2–3pm
Wed 3–4pm

Recall: Descartes' Big Idea.

a point $\equiv$ an ordered list of numbers.
(the "cartesian coordinates" of the point)

Last time we talked about "adding" points (i.e. vector addition)

This time we'll try to "multiply" points.

$\mathbb{R}$ = the set of "real" numbers
= the "number line".

Let

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}.$$

= ordered n-tuples of real numbers
= n-dimensional space.

We can multiply elements of $\mathbb{R}^1 = \mathbb{R}$.

We can also multiply elements of $\mathbb{R}^2$:

Given (u_1, u_2) and (v_1, v_2) define

$$(u_1, u_2) * (v_1, v_2) := (u_1 v_1 - u_2 v_2, u_1 v_2 + u_2 v_1).$$

Do you recognize this??

Think: (u_1, u_2) "=" $u_1 + u_2\sqrt{-1}$
(v_1, v_2) "=" $v_1 + v_2\sqrt{-1}$

Then $(u_1, u_2) * (v_1, v_2)$

$$\text{"="} (u_1 + u_2\sqrt{-1})(v_1 + v_2\sqrt{-1})$$
$$= u_1 v_1 + u_1 v_2\sqrt{-1} + u_2 v_1\sqrt{-1} + u_2 v_2 \underbrace{\sqrt{-1}\sqrt{-1}}_{-1}.$$

$$= (u_1 v_1 - u_2 v_2) + (u_1 v_2 + u_2 v_1)\sqrt{-1}.$$

$$\text{"="} (u_1 v_1 - u_2 v_2, u_1 v_2 + u_2 v_1).$$

We call this

$\mathbb{R}^2 = \mathbb{C}$ "complex numbers".

Hard Fact:

It is NOT POSSIBLE to "multiply" elements of $\mathbb{R}^n$ except when $n = 1$ or 2.

Oh well . . . .

So we can't mutiply

point * point = point.

But there is a nice way to define

point * point = number.

Define the dot product of vectors

$\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$

by

$$\boxed{\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.}$$

Define the length of the vector $\vec{u}$ by.

$$\boxed{\|\vec{u}\|^2 := \vec{u} \circ \vec{u}}$$

$$= u_1^2 + u_2^2 + \cdots + u_n^2$$

i.e. $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$

[Note that this is a definition, NOT a theorem.]

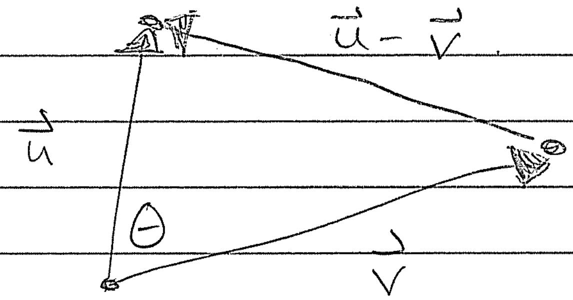
Exercise: Show that vector addition and dot product behave well together.

For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ we have

$$\vec{u} \circ (\vec{v} + \vec{w}) = (\vec{u} \circ \vec{v}) + (\vec{u} \circ \vec{w}).$$

"distributive law"

Now let $\vec{u}, \vec{v} \in \mathbb{R}^n$ and consider the triangle



(This is a 2-dim triangle in n-dim space)

① Arithmetic Says:

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v}) \circ (\vec{u}-\vec{v})$$

$$= \vec{u} \circ \vec{u} - \vec{u} \circ \vec{v} - \vec{v} \circ \vec{u} + \vec{v} \circ \vec{v}.$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u} \circ \vec{v}).$$

② Geometry Says (Law of Cosines):

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \cdot \|\vec{v}\| \cos\theta$$

Comparing ① & ② tells us

$$\boxed{\vec{u} \circ \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos\theta}.$$

This is beautiful!

Thus the dot product allows us to define distances

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

$$= \sqrt{(\vec{u} - \vec{v}) \circ (\vec{u} - \vec{v})}$$

and angles

$$\cos\theta = \frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{\vec{u} \circ \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}\right)$$

$\mathbb{R}^n$ with $+$ and $\circ$ is called n-dimensional "Euclidean space"

This is the modern DEFINITION of space!