Math 230 E **Fall 2013**
Homework 5 **Drew Armstrong**

Problem 1. Use induction to prove that for all integers $n \geq 1$ we have

$$\text{“}1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.\text{”}$$

This result appears in the *Aryabhatiya* of Aryabhata (499 CE, when he was 23 years old). [Hint: You may assume the result $1 + 2 + \cdots + n = n(n + 1)/2$.]

Problem 2. Recall that $a \equiv b \pmod{n}$ means that $n|(a - b)$. Use induction to prove that for all $n \geq 2$, the following holds:

“**if** $a_1, a_2, \ldots, a_n \in \mathbb{Z}$ such that each $a_i \equiv 1 \pmod 4$, **then** $a_1 a_2 \cdots a_n \equiv 1 \pmod 4$.”

[Hint: Call the statement $P(n)$. Note that $P(n)$ is a statement about **all** collections of n inegers. Therefore, when proving $P(k) \Rightarrow P(k+1)$ you must say “Assume that $P(k) = T$ and consider any $a_1, a_2, \ldots, a_{k+1} \in \mathbb{Z}$.” What is the base case?]

Problem 3 (Generalization of Euclid's Proof of Infinite Primes)

(a) Consider an integer $n > 1$. **Prove** that if $n \equiv 3 \pmod 4$ then n has a prime factor of the form $p \equiv 3 \pmod 4$. [Hint: You may assume that n has a prime factor p, which we proved in class. Note that there are three kinds of primes: the number 2, primes $p \equiv 1 \bmod 4$ and primes $p \equiv 3 \bmod 4$. Use Problem 2.]

(b) Prove that there are infinitely many prime numbers of the form $p \equiv 3 \pmod 4$. [Hint: Assume there are only **finitely** many and call them $3 < p_1 < p_2 < \cdots < p_k$. Then consider the number $N := 4p_1 p_2 \cdots p_k + 3$. By part (a) this N has a prime factor of the form $p \equiv 3 \pmod 4$. Show that this p is not in the list. Contradiction.]

Problem 4. Consider the following two statements/principles.

WO: Every nonempty subset $S \subseteq \mathbb{N} = \{1, 2, 3, \ldots\}$ has a least element.

PI: **If** $P : \mathbb{N} \to \{T, F\}$ is a family of statements satisfying

- $P(1) = T$ and
- for any $k \geq 1$ we have $P(k) \Rightarrow P(k + 1)$,

then $P(n) = T$ for all $n \in \mathbb{N}$.

Now **prove** that WO $\Rightarrow$ PI. [Hint: Assume WO and assume that $P : \mathbb{N} \to \{T, F\}$ is a family of statements satisfying the hypotheses of PI. You want to show that $P(n) = T$ for all $n \geq 1$. Assume for contradiction that there exists $n \geq 1$ such that $P(n) = F$ and let S be the set of numbers $n \geq 1$ such that $P(n) = F$. By WO the set S has a least element $m \in S$. Since $P(1) = T$ we must have $m \geq 2$. Now use the existence m to derive a contradiction. Hence $P(n) = T$ for all $n \geq 1$ and it follows that PI is true.]