Math 230 E **Fall 2013**
Homework 3 **Drew Armstrong**

Problem 1. Let X and Y be **finite** sets.

(a) If there exists a **surjective** function $f : X \to Y$, prove that $|X| \geq |Y|$.
(b) If there exists an **injective** function $g : X \to Y$, prove that $|X| \leq |Y|$.
(c) If there exists a **bijective** function $h : X \to Y$, prove that $|X| = |Y|$.

[Hint: For parts (a) and (b), for each $y \in Y$ let $d(y)$ be the number of arrows pointing to $y \in Y$. What happens if you sum the numbers $d(y)$ for all $y \in Y$? Recall the **definitions** from the course notes.]

Problem 2. For all integers $a, b \in \mathbb{Z}$ with $b \neq 0$, we define an **abstract symbol** "$\frac{a}{b}$". We declare rules for "multiplying" and "adding" abstract symbols,

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd} \qquad \text{and} \qquad \frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd},$$

and we declare that the abstract symbols $\frac{a}{b}$ and $\frac{c}{d}$ are "equal" if and only if $ad = bc$. Let $\mathbb{Q}$ denote the set of abstract symbols (we call this the system of **rational numbers**). For all rational numbers $x \in \mathbb{Q}$, prove that x can be expressed as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ have no common divisor except ± 1. (We say that the fraction x can be written in "lowest terms".) [Hint: Let S be the set of absolute values of all the possible numerators of x:

$$S := \left\{ |a| \in \mathbb{N} : \exists\, a, b \in \mathbb{Z} \text{ such that } x = \frac{a}{b} \right\} \subseteq \mathbb{N}.$$

Since $x \in \mathbb{Q}$, the set S is not empty, so by Well-Ordering it has a smallest element.]

Problem 3. The Division Algorithm 2.12 says that for all $a, b \in \mathbb{Z}$ with $b > 0$ *there exist unique* $q, r \in \mathbb{Z}$ such that $a = qb + r$ and $0 \leq r < b$. Explicitly use this to prove the following: For all $a, b \in \mathbb{Z}$ with $b > 0$ *there exists a unique* integer $k \in \mathbb{Z}$ such that

$$k \leq \frac{a}{b} < k + 1.$$

[Note: You must prove both the *existence* and the *uniqueness* of k. Don't be a hero; **quote** the Division Algorithm. You do not need to reduce everything to the axioms.]

Problem 4. How do $-$ and $\times$ interact? Prove the following exercises using the axioms of $\mathbb{Z}$ from the handout. It will save time if you assume the Cancellation Property that was proved on the previous homework: $\forall\, a, b, c \in \mathbb{Z}$, $(a + b = a + c) \Rightarrow (b = c)$.

(a) Prove that for all $a \in \mathbb{Z}$ we have $0a = 0$.
(b) Recall that $-n$ is the unique integer such that $n + (-n) = 0$. Prove that for all $a, b \in \mathbb{Z}$ we have $(-a)b = -(ab)$. [Hint: You will need part (a).]
(c) Prove that for all $a, b, c \in \mathbb{Z}$ we have $a(b - c) = ab - ac$. [Hint: Use part (b).]
(d) Prove that for all $a, b \in \mathbb{Z}$ we have $(-a)(-b) = ab$. [Hint: Use part (a) to show that $ab + a(-b) = 0$ and then use part (b). Note that $-(-n) = n$ for all $n \in \mathbb{Z}$.]

[Now if a child asks you **why** negative times negative is positive, you will know what to say.]