Fri Sept 7

HW 1 due NOW

Q: What is "space"?
What is a "point"?
What is a "number"?

Pythagoras:
- "number" = a ratio of whole numbers.
- But $\sqrt{2}$ is not a "number" OOPS!

⇝ Math was founded on points, lines, circles, NOT numbers.

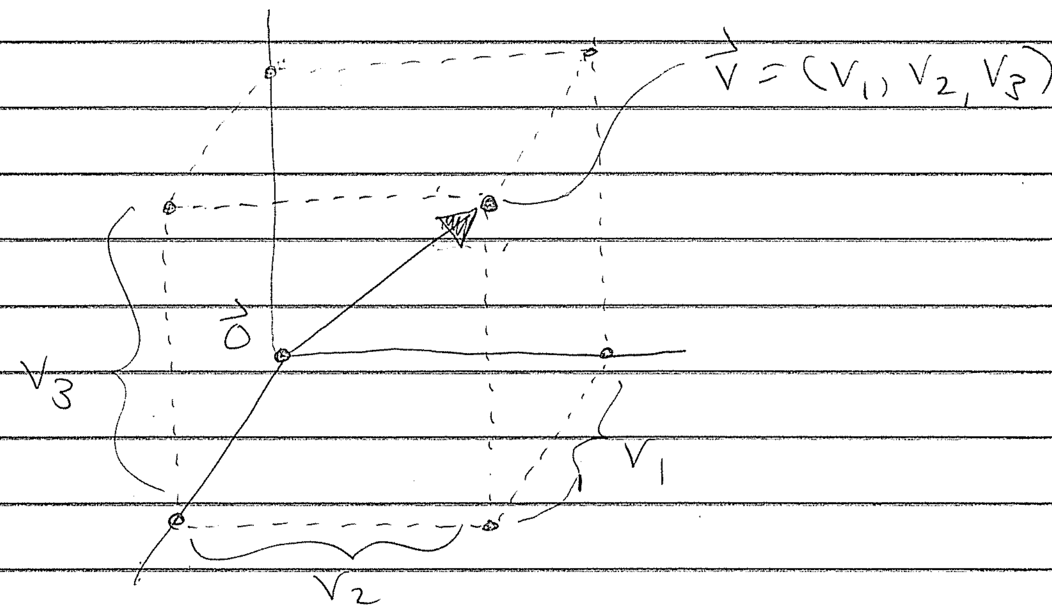
Euclid:
- a "point" is that which has no part
- a "line" is breadthless length.
- a "number" is the length of a line segment.

The number/geometry rift persisted for ~1900 years!

Then: Descartes (1596 – 1650)

"La Géométrie" (1637)
- invented "coordinate geometry".

Descartes' new idea: (a fly in the corner)
- a "point" is a list of numbers.



- fix 3 perpendicular axes
- given a point $\vec{v}$ draw a <u>rect</u>angular box between $\vec{0}$ and $\vec{v}$.
- if the dimensions are v_1, v_2, v_3 we say

$$\vec{v} = (v_1, v_2, v_3)$$

the "Cartesian coordinates" of the point.

- also think: $\vec{v}$ is an arrow ("vector") with tail at $\vec{0} = (0,0,0)$.

HW 1.4: The length of the vector is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^3}$$

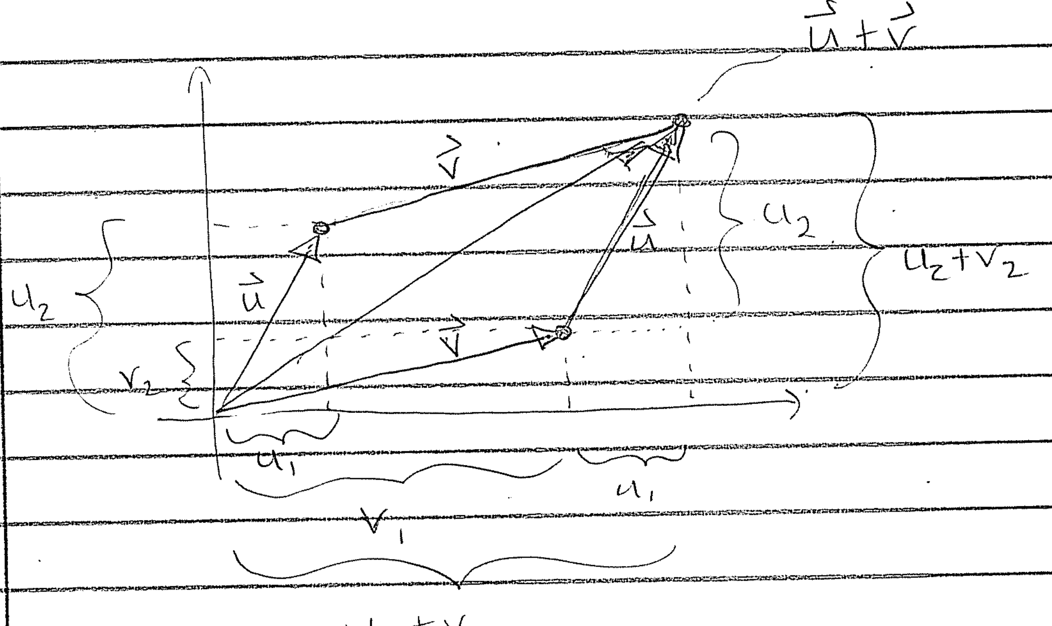
So what?

Here's something <u>new</u>: We can add "points"

$$\vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3)$$
$$:= (u_1 + v_1, u_2 + v_2, u_3 + v_3).$$

(What would Euclid think?!)

Picture (in 2D): $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$



Parallelogram Law: vectors add "head-to-tail"

Subtraction: What is $\vec{u}-\vec{v}$?

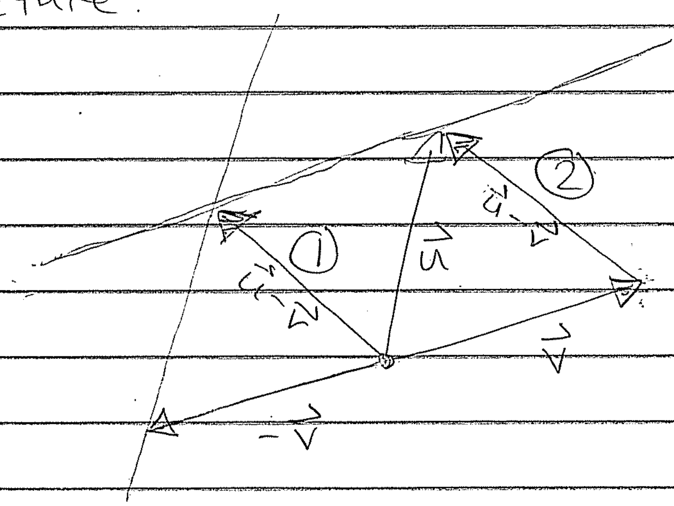
Two possibilities.

same length
opposite direction

① $\vec{u}-\vec{v} = \vec{u} + "-\vec{v}"$

② $\vec{u}-\vec{v}$ is the vector $\vec{x}$ that solves
$\vec{v}+\vec{x}=\vec{u}$.

Picture.



① & ② are the same vector!

Hence we can say

$$\mathrm{dist}(\vec{u},\vec{v}) = \|\vec{u}-\vec{v}\|$$

$$= \sqrt{(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2}$$

~~proof HW 1.4~~

Q: Given points $\vec{u}, \vec{v}$ in 4D space, is it true that

$$\text{dist}(\vec{u}, \vec{v}) = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2 + (u_4-v_4)^2} \;?$$

How could you prove it?

A: No one had the guts to ask this until ~ 1850.

Modern Definition of "space":

Let $\mathbb{R}^n$ be the set of vectors
$\vec{v} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$

For $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$, define the dot product

$$\vec{u} \cdot \vec{v} := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Define the length $\|\vec{u}\|$ of a vector $\vec{u}$

$$\|\vec{u}\|^2 := \vec{u} \cdot \vec{u}$$
$$= u_1^2 + u_2^2 + \cdots + u_n^2.$$

and define the distance between two points

$$\operatorname{dist}(\vec{u}, \vec{v}) := \| \vec{u} - \vec{v} \|$$

$$= \sqrt{(\vec{u} - \vec{v}) \circ (\vec{u} - \vec{v})}$$

Upshot:

The "Pythagorean Theorem" is no longer a Theorem.
It's the <u>definition</u> of "space".

"Pythagorean Theorem" = "dot product of vectors"

Mon Sept 10.

HW 2 due next Wed Sept 19.
Exam 1 on Fri Sept 21.

Right Now: The transition

Euclid (~300 BC) $\rightsquigarrow$ Descartes (1637)
"points, lines circles" "numbers"

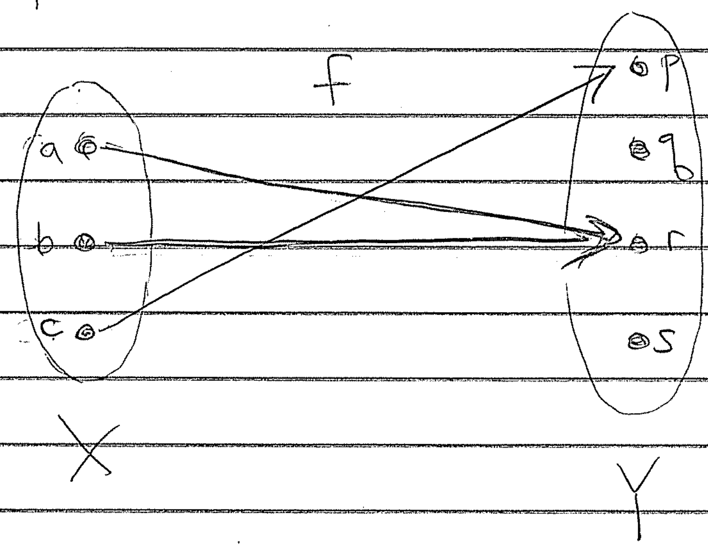
Discussion: Some Logic.

Let X and Y be sets (collections of things). A <u>function</u> $f: X \to Y$ is a collection of arrows satisfying two rules

(F1) Every arrow points from an element of X to an element of Y.

(F2) Every element of X has <u>exactly one</u> <u>arrow</u> pointing from it

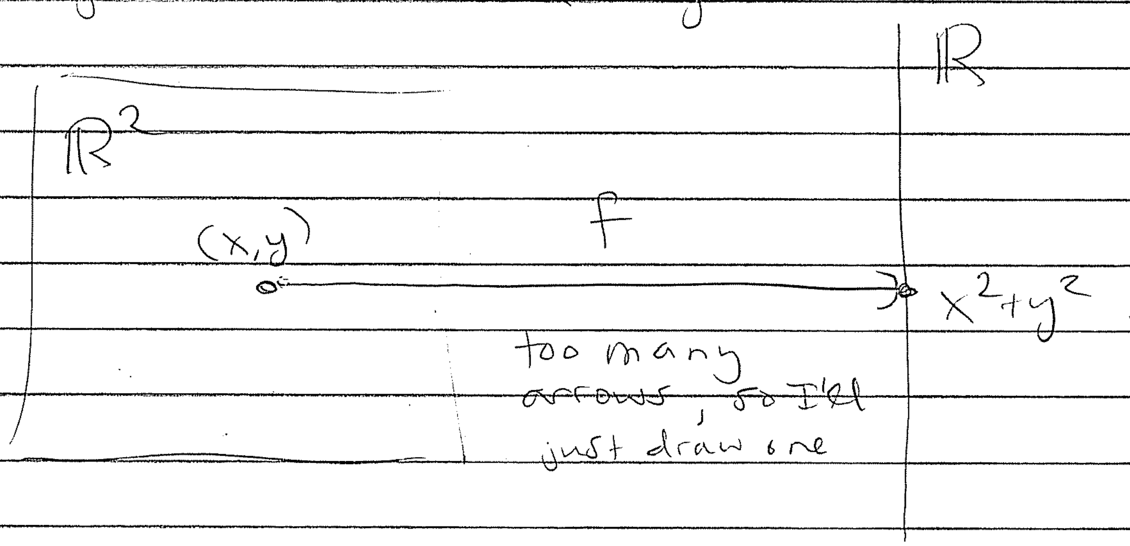
Example:



Example:
Sometimes we define a function by a formula. For example, define $f: \mathbb{R}^2 \to \mathbb{R}$ by the formula.

$$f(x,y) := x^2 + y^2.$$

To each point (x,y) in the plane $\mathbb{R}^2$, assign the number x^2+y^2

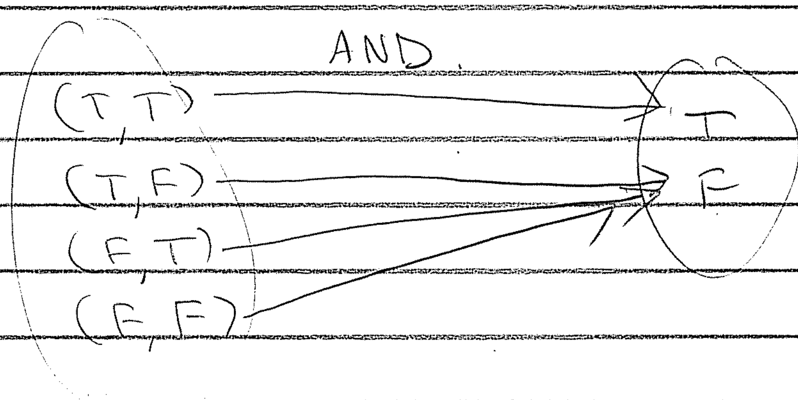


Example: Consider the set $\{T, F\}$ and the set of ordered pairs

$$\{T,F\}^2 := \{(T,T), (T,F), (F,T), (F,F)\}$$

Then we define 3 important functions

(1) $\text{AND}: \{T,F\}^2 \to \{T,F\}$



We can also write this with a "formula"

$$\left.\begin{array}{l} T \text{ AND } T = T \\ T \text{ AND } F = F \\ F \text{ AND } T = F \\ F \text{ AND } F = F \end{array}\right\} \quad \text{In general} \quad (P,Q) \longmapsto P \text{ AND } Q.$$

(2) OR : $\{T, F\}^2 \to \{T, F\}$

$$
\begin{aligned}
(T,T) &\xrightarrow{\text{OR}} T \\
(T,F) &\xrightarrow{\text{OR}} T \\
(F,T) &\xrightarrow{\text{OR}} T \\
(F,F) &\xrightarrow{\text{OR}} F
\end{aligned}
$$

In general $(P, Q) \mapsto P \text{ OR } Q$

(3) NOT : $\{T, F\} \to \{T, F\}$

$$
\begin{aligned}
T &\xrightarrow{\text{NOT}} F \\
F &\xrightarrow{\text{NOT}} T
\end{aligned}
$$

In general: $P \mapsto \text{NOT } P$.

We can compose AND, OR, NOT to get lots of functions from $\{T, F\}^n$ to $\{T, F\}$

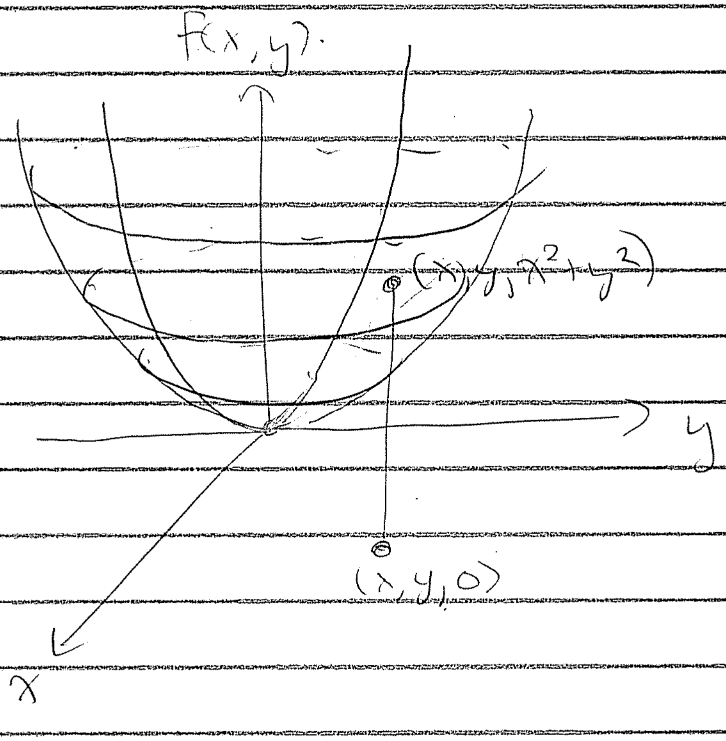
e.g. $(P, Q, R) \mapsto \text{NOT}(P \text{ OR } (Q \text{ AND } (P \text{ OR } R)))$

These are called "Boolean functions" after George Boole (1815 – 1864).

– "An Investigation of the Laws of Thought" (1854).

Sometimes we like to describe a function by drawing its "graph" (some picture of the collection of arrows.

Example: If we think of the arrow $(x,y) \longrightarrow x^2+y^2$ as the triple (x,y,x^2+y^2) in 3D space $\mathbb{R}^3$, then the collection of arrows looks like.



the graph is a "paraboloid"

Q: What is the "graph" of a Boolean function?

Answer: A "truth table"

P	Q	P AND Q	NOT (P AND Q)
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

the domain is the "P,Q-plane"

this is the "graph" of P AND Q

Application:

P	Q	NOT P	NOT Q	(NOT P) OR (NOT Q)
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

We conclude

NOT (P AND Q) = (NOT P) OR (NOT Q)

the same function

Wed Sept 12

HW 1 solutions on web. (Ave 15.6 Med 16 / 20)
HW 2 due Wed Sept 19
Exam 1 on Fri Sept 21
Office Hours today 3 – 4.

[Discussion Continued ... Truth Tables.

Recall: If X is a set of things, then

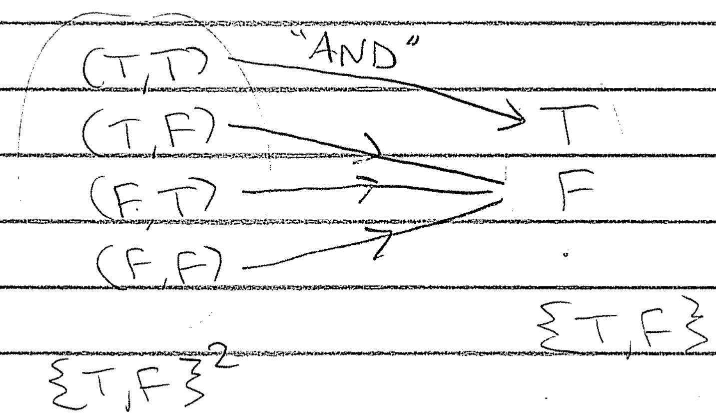
$$X^n = \{ (x_1, x_2, \cdots, x_n) : \text{each } x_i \in X \}$$

is the set of ordered n-tuples of elements.
(e.g. Cartesian coordinates $\mathbb{R}^n$).

e.g. $\{T,F\}^3 = \{ (T,T,T), (T,T,F), (T,F,T), (T,F,F), (F,T,T), (F,T,F), (F,F,T), (F,F,F) \}$

Recall: A "Boolean function" is any function from $\{T,F\}^n$ to $\{T,F\}$

e.g.



Our favorite picture of a Boolean function is a Truth Table:

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

FACT (not too hard): <u>Any</u> Boolean function has a "formula" in terms of <u>3 basic functions</u>

AND : $\{T,F\}^2 \to \{T,F\}$
OR : $\{T,F\}^2 \to \{T,F\}$
NOT : $\{T,F\} \to \{T,F\}$

Example: Here's a Boolean function

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F (only $T \Rightarrow F$ is BAD).
F	T	T
F	F	T

Problem: Give a formula for $\Rightarrow$ in terms of AND, OR, NOT.

Claim: For all $P, Q \in \{T, F\}$ (we say P, Q are "Boolean variables") we have

$$(P \Rightarrow Q) = ((\text{NOT } P) \text{ OR } Q).$$

~~the same function~~

Proof:

P	Q	NOT P	(NOT P) OR Q	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$\boxtimes$

WARNING: Boolean formulas are not unique!

Exercise: Show that

$$(P \Rightarrow Q) = \Big((P \text{ AND } Q) \text{ OR } ((\text{NOT } P) \text{ AND } Q) \text{ OR } ((\text{NOT } P) \text{ AND } (\text{NOT } Q))\Big)$$

[Hint: Rows of the Truth Table.]

Definition: If two Boolean functions are equal (as functions), we say they are *logically equivalent*.

Example: For all $P, Q \in \{T, F\}$ we have

$$\text{NOT}(P \text{ OR } Q) = (\text{NOT } P) \text{ AND } (\text{NOT } Q)$$
$$\text{NOT}(P \text{ AND } Q) = (\text{NOT } P) \text{ OR } (\text{NOT } Q).$$

"de Morgan's Laws".

Proof: Exercise.

Instead, I'll prove that
$$(P \Rightarrow (Q \text{ OR } R)) = ((P \text{ AND } (\text{NOT } Q)) \Rightarrow R)$$

Proof:

P	Q	R	Q OR R	P⇒(Q OR R)	NOT Q	P AND NOT Q	(P AND NOT Q) ⇒ R
T	T	T	T	T	F	F	T
T	T	F	T	T	F	F	T
T	F	T	T	T	T	T	T
T	F	F	F	F	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	F	T	T	F	T

$\boxtimes$

Fri Sept 14.

HW 2 due Wed Sept 19

Exam 1 Fri Sept 21.

(look at practice exam — please ignore problem 1; I didn't teach you that stuff)

[Discussion Continued:

Logical Equivalence.

Recall that for all $P, Q \in \{T, F\}$ we have

$$\text{NOT}(P \text{ OR } Q) = (\text{NOT } P) \text{ AND } (\text{NOT } Q)$$
$$\text{NOT}(P \text{ AND } Q) = (\text{NOT } P) \text{ OR } (\text{NOT } Q)$$

"de Morgan's Laws"

and for all $P, Q, R \in \{T, F\}$ we have.

$$(P \Rightarrow (Q \text{ OR } R)) = ((P \text{ AND } (\text{NOT } Q)) \Rightarrow R)$$

But this is not a course about logic. Why do we care?

$\int$

Application: Let m, n be integers. If m^3+n^3 is odd, prove that either m or n (or both) must be odd.

(Note: OR is inclusive, i.e. P OR Q means "P or Q or both". If we mean "P or Q but not both" we write the exclusive or P XOR Q.)

So let P = "m^3+n^3 is odd"
Q = "m is odd"
R = "n is odd"

Want to prove that $P \Rightarrow (Q \text{ OR } R)$.

Here are two proofs.

Proof 1: We will show the contrapositive statement NOT(Q OR R) $\Rightarrow$ NOT P
i.e. (NOT Q) AND (NOT R) $\Rightarrow$ NOT P
i.e. "m is even and n is even" $\Rightarrow$ "m^3+n^3 is even".

So suppose m and n are both even, say $m = 2k$ and $n = 2l$. Then we have

$$\begin{aligned} m^3 + n^3 &= (2k)^3 + (2l)^3 \\ &= 8k^3 + 8l^3 \\ &= 2(4k^3 + 4l^3), \end{aligned}$$

which is even, as desired. $\blacksquare$

Proof 2: We will show the equivalent statement

$$(P \text{ AND } (\text{NOT } Q)) \Rightarrow R$$

i.e. "$m^3 + n^3$ is odd and m is even" $\Rightarrow$ "n is odd"

So suppose $m^3 + n^3$ is odd, say $m^3 + n^3 = 2k+1$, and suppose m is even, say $m = 2l$. Then we have

$$\begin{aligned} n^3 &= (m^3 + n^3) - m^3 \\ &= 2k+1 - (2l)^3 \\ &= 2k+1 - 8l^3 \\ &= 2(k - 4l^3) + 1, \end{aligned}$$

which is odd. It is easy to see that "n even $\Rightarrow n^3$ even" so the contrapositive "n^3 odd $\Rightarrow n$ odd" is also true. We conclude that n is odd, as desired. $\blacksquare$

Let X and Y be sets.

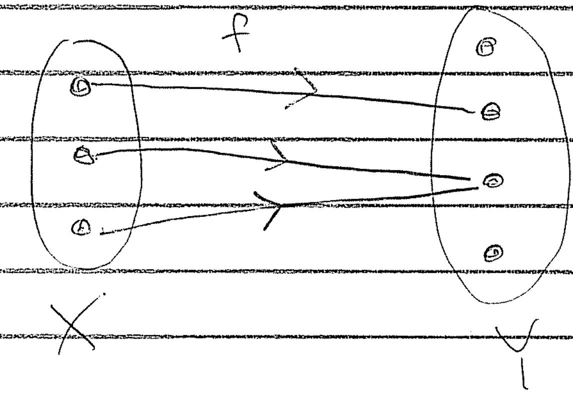
Recall: A function $f : X \to Y$ is a collection of arrows from X to Y satisfying

(F1) Every $x \in X$ has ≥ 1 arrow

(F2) Every $x \in X$ has ≤ 1 arrow.

(Together F1 & F2 say: Every $x \in X$ has exactly 1 arrow).

eg.



Special kinds of functions:

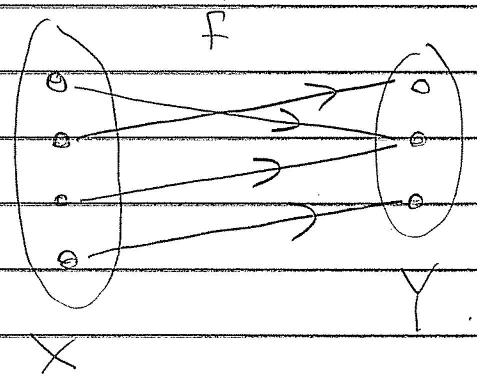
If function $f : X \to Y$ satisfies

(F3) Every $y \in Y$ has ≥ 1 arrow,

We say f is ONTO Y

or f is "surjective"

eg



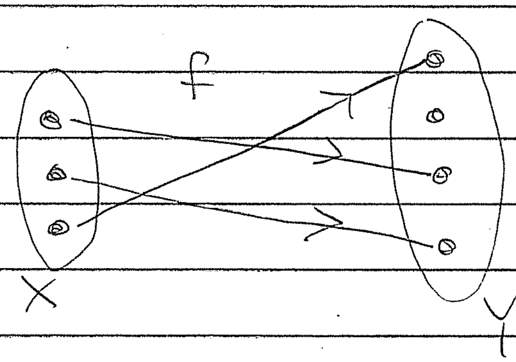
is surjective.

If function $f: X \to Y$ satisfies

(F4) Every $y \in Y$ has ≤ 1 arrow,

we say f is 1-to-1 (or 1:1)
or f is "injective"

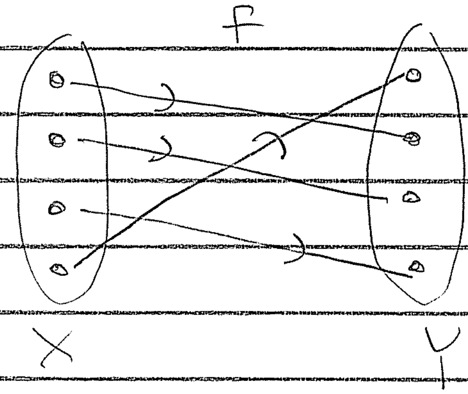
eg



is injective.

If function $f: X \to Y$ satisfies
(F3) and (F4), i.e. every $y \in Y$ has exactly one arrow then we say
f is a "1:1 correspondence"
or f is "bijective"
or (a "bijection")

eg.



is a bijection

"Thinking Homework": let X, Y be finite sets.

(1) If f is surjective then $|X| \geq |Y|$
(2) If f is injective then $|X| \leq |Y|$
(3) If f is bijective then $|X| = |Y|$.

Mon Sept 17

HW 2 due this Wed
Exam 1 this Fri

New course notes are online with "proof 2" corrected.
Assignment: Read them. AND read Section 1.5 of the text.

[. . . . end of logical discussion.

Let X and Y be sets. Let f be a collection of arrows from X to Y. Let f^{-1} be the collection of "reversed" arrows from Y to X.

Consider four properties

F1: Each $x \in X$ has ≥ 1 arrow.
F2: Each $x \in X$ has ≤ 1 arrow.
F3: Each $y \in Y$ has ≥ 1 arrow.
F4: Each $y \in Y$ has ≤ 1 arrow.

DEF: If f satisfies F1 and F2 we say f is a function and we write

$$f : X \to Y.$$

"f is a function from X to Y"

F1 + F2 + F3 : $f: X \to Y$ is surjective

F1 + F2 + F4 : $f: X \to Y$ is injective

F1 + F2 + F3 + F4 : $f: X \to Y$ is bijective

Note: f^{-1} is a function $\iff$ F3 and F4 hold.

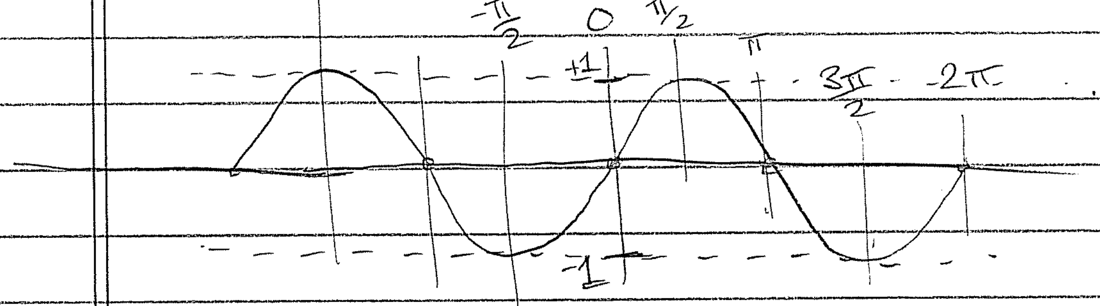
Conclusion ("Inversion Theorem", pg. 136):

A function $f: X \to Y$ has an inverse $f^{-1}: Y \to X$ (say f is "invertible")

$\Updownarrow$

f is a bijection.

Example: Is the function $\sin : \mathbb{R} \to \mathbb{R}$ invertible? NO.



$\sin : \mathbb{R} \to \mathbb{R}$ is NOT surjective!

NOT injective!

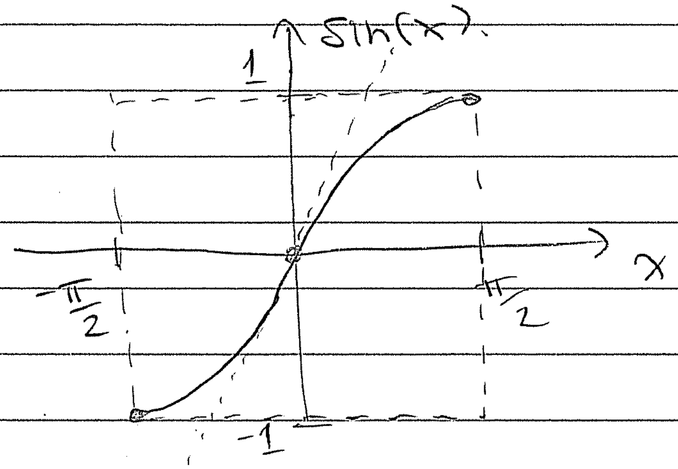
(so $\sin^{-1}$ is not a function).

What can we do?

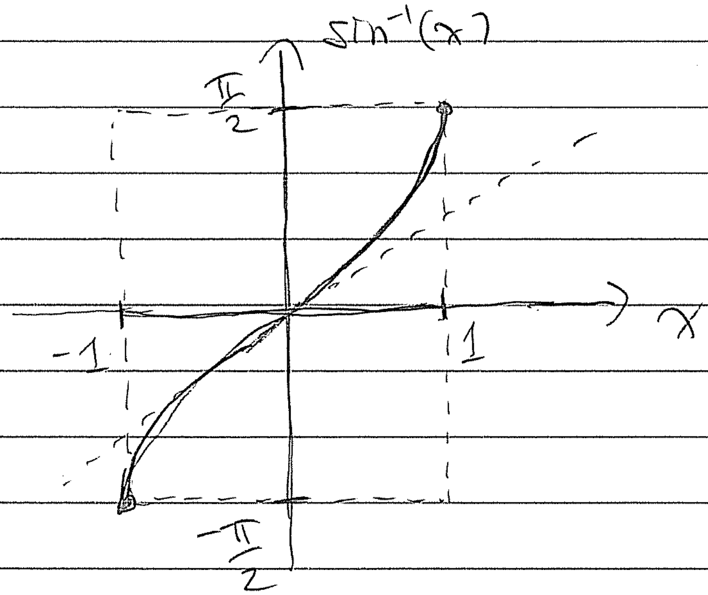
Restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Restrict the codomain to $[-1, 1]$

Graph of $\sin : \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \to [-1, 1]$



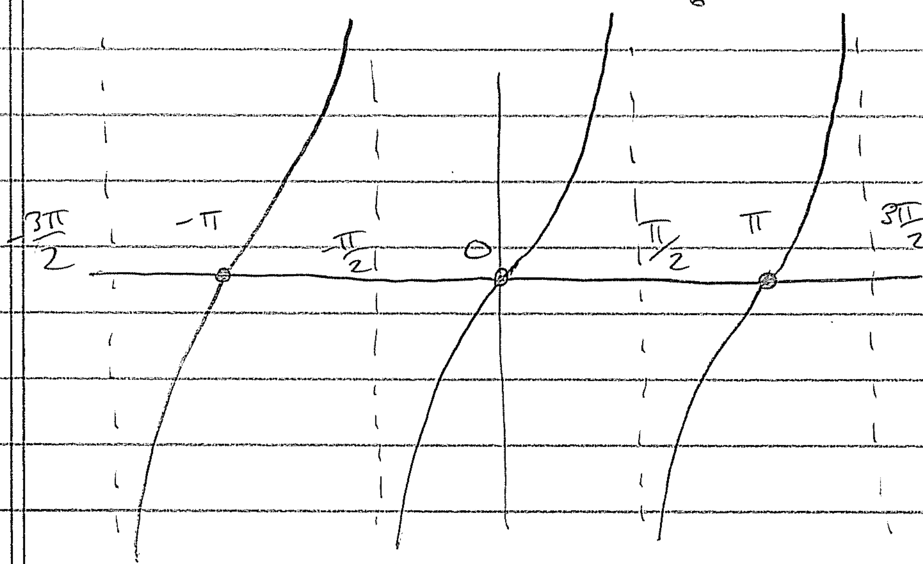
Now sin is a bijection, so it has an inverse function $\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Example: Is the function $\tan : \mathbb{R} \to \mathbb{R}$ invertible? **NO**

actually NOT a function, so we replace the domain by $\mathbb{R} - \{\pi + \frac{k\pi}{2}$ for all $k\}$

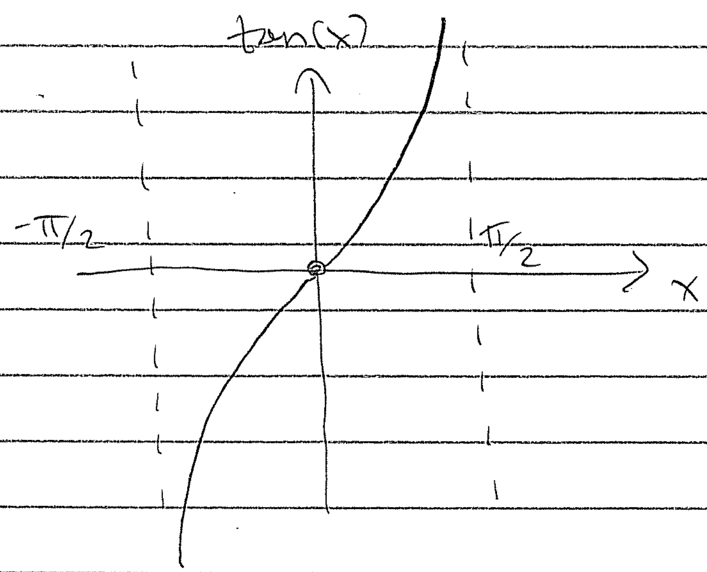
call this $\hat{\mathbb{R}}$



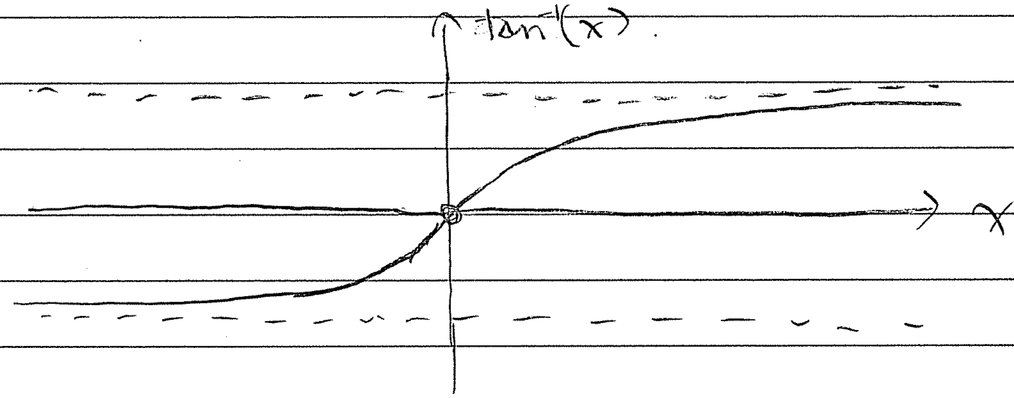
$\tan : \hat{\mathbb{R}} \to \mathbb{R}$ IS surjective but not injective.

What to do? Restrict the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Then $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ is a bijection



Hence it has an inverse function
$\tan^{-1}: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



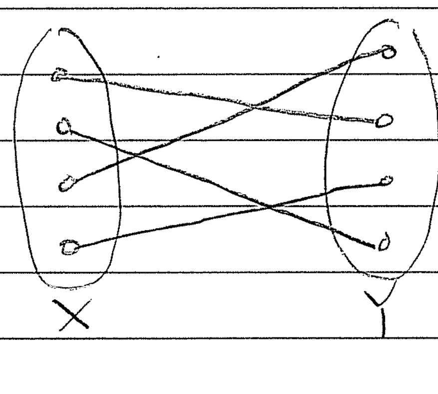
Strange Observation: There is a 1:1 correspondence (i.e. bijection) between all real numbers $\mathbb{R}$ and numbers in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Why strange?
Because: Let X, Y be <u>FINITE</u> sets and let $f: X \to Y$ be a function. Then

$$\left.\begin{array}{ll} (1) & f \text{ surjective} \implies |X| \geq |Y| \\ (2) & f \text{ injective} \implies |X| \leq |Y|. \end{array}\right\} \text{Think: Why?}$$

$(1)+(2) =$ (3) f bijective $\implies |X| = |Y|$

e.g.



the elements are paired up.

What about infinite sets?

Since $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection, can we say

$$|(-\tfrac{\pi}{2}, \tfrac{\pi}{2})| = |\mathbb{R}| \; ?$$

YES, If you want to.
(Georg Cantor 1845 – 1918).

Cantor DEFINED two infinite sets to have the same "size" (or "cardinality") if there exists a bijection between them.
"$\exists$"

Q: So not all infinite sets have the same size?

A: Correct.
In fact, $|\mathbb{Q}| < |\mathbb{R}|$

fractions
"rational numbers"

end of discussion.]

Wed Sept 19

HW 2 due NOW.
Exam 1 Friday
- closed book.
- no cheating

Today: Review.

HW 2 Problem 4. Let m, n be integers. Prove that

"m is even OR n is even" $\iff$ "mn is even".

Proof: First let P = "m is even", Q = "n is even", R = "mn is even". We want to prove that $(P \text{ OR } Q) \iff R$

First we will prove $(P \text{ OR } Q) \Rightarrow R$. By Problem 3(b) this is equivalent to $(P \Rightarrow R)$ AND $(Q \Rightarrow R)$. To show $P \Rightarrow R$, suppose m is even, say $m = 2k$. Then $mn = 2kn$ is even. To show $Q \Rightarrow R$ suppose n is even, say $n = 2l$. Then $mn = m2l = 2(ml)$ is even.

[Or if you want to think...

We only need to show that

$$\underset{T}{(P \text{ OR } Q)} \Rightarrow \underset{F}{R}$$

doesn't happen. If $(P \text{ OR } Q) = T$,
there are 3 cases.

1. $(P, Q) = (T, T)$
2. $(P, Q) = (T, F)$
3. $(P, Q) = (F, T)$.

Check that none of these allow $R = F$.
(Actually only need to check 2 and 3,
which is exactly what we did.)

]

Next we will prove $R \Rightarrow (P \text{ OR } Q)$.
By Problem 3(a) this is equivalent to
$(R \text{ AND NOT } P) \Rightarrow Q$
So suppose that mn is even and m is odd.
In this case, if n were odd we would
have mn odd by Problem 1(a). Contradiction.
Hence n is even. $\blacksquare$

[More thinking . . . .

We need to show $\underset{T}{R} \Rightarrow \underset{F}{(P \text{ OR } Q)}$ doesn't happen.

How could it? If $(P \text{ OR } Q) = F$ then $(P, Q) = (F, F)$. So there is just one case: Show that $(P, Q, R) = (F, F, T)$ is impossible. How? Assume it's possible and get a contradiction. How? There are two possible ways.

Way ① Show that $(P, R) = (F, T) \Rightarrow Q \neq F$.
which we <u>did</u>.

Way ② Show that $(P, Q) = (F, F) \Rightarrow R \neq T$.
$\downarrow$

Do it: To prove $R \Rightarrow (P \text{ OR } Q)$ we will show the contrapositive $(\text{NOT } P \text{ AND NOT } Q) \Rightarrow \text{NOT } R$.
So suppose m and n are even, say $m = 2k$ and $n = 2l$. Then $mn = 2k\,2l$
$= 2(2kl)$ is even

again.

Q: Are you expected to come up with new logical tricks every time?

A: NO. We (mathematicians) only use a handful of logical tricks. READ Chapter 1.5.

=

So what is "logic"? Never mind!

For us, there are just two axioms.

(L1) Excluded middle: Every mathematical statement is either T or F. NOT both. NOT neither.

(L2) Implication: $P \Rightarrow Q$ means exactly that $(P,Q) = (T,F)$ can't happen. (T flows along arrows).

(L2') Contrapositive: F flows backwords

(Note L2 and L2' mean the same)

It is also convenient to use the symbols AND, OR, NOT.

Useful remarks:

- Logical equivalence is proved with a truth table.

- de Morgan: $\forall P, Q \in \{T, F\}$

$$\text{NOT } (P \text{ OR } Q) = (\text{NOT } P) \text{ AND } (\text{NOT } Q)$$
$$\text{NOT } (P \text{ AND } Q) = (\text{NOT } P) \text{ OR } (\text{NOT } Q).$$

- $\forall P, Q \in \{T, F\}$, $P \Rightarrow Q = (\text{NOT } P) \text{ OR } Q$ (*)

From Practice Exam 1, replace P by NOT P to get $(\text{NOT } P) \Rightarrow Q = P \text{ OR } Q$.

Replace Q by NOT Q in (*) to get
$(\text{NOT } P) \text{ OR } (\text{NOT } Q) = P \Rightarrow (\text{NOT } Q)$.

Then use de Morgan to get

$$P \text{ AND } Q = \text{NOT } \big(P \Rightarrow (\text{NOT } Q)\big).$$