Wed Aug 22

Memorial 215

MTH 230

Intro to "Abstract Mathematics"

(A.K.A. Mathematics)

Drew Armstrong

Ungar 533

Web: www.math.miami.edu/~armstrong

Office Hours: TBA

Evaluation:

probably 6
usually due
on Fridays

	Homework	25%
in class	Exam 1	25%
	Exam 2	25%
	Exam 3	25%
		100%

NO FINAL EXAM

Topic of MTH 280 : Math.

Q: What is Mathematics?
(submit your answer on a note card)

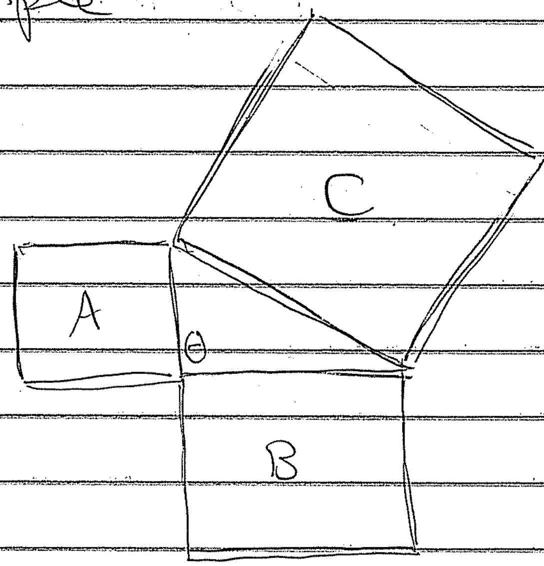
My Answer:
- math allows humans to agree on things.
- it's a language but NOT a "natural language"
- deliberately invented to be

CLEAR & PRECISE.

"Rigor = Clarity + Precision"

- it uses English words but this is deceptive
- learn it like a foreign language.
  - immersion
  - practice
  - patience
  - OPEN MIND.

Example:



Claim: If $\theta = 90^\circ$

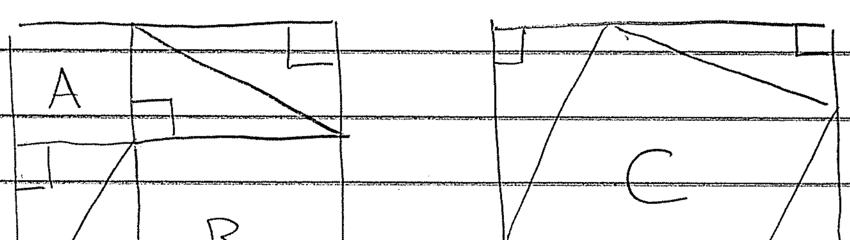
Then area(C) = area(A) + area(B) ///

Why is this "true"?

Is it "true"?

I'll try to convince you.

Proof: Assume that $\theta = 90^\circ$. Then observe the following two squares

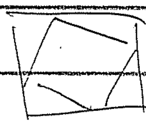


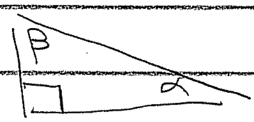
They have equal area.
Each contains 4 of the original triangles.
Remove the triangles.
What remains must have equal area.
Hence

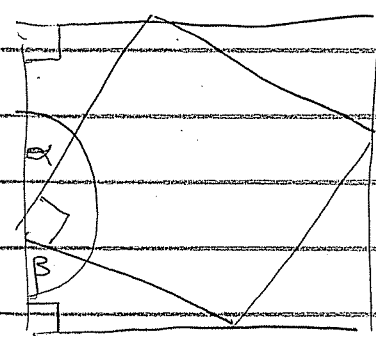
area(A) + area(B) = area(C)

Are you convinced?

Possible Complaints:

- Why is  a square?

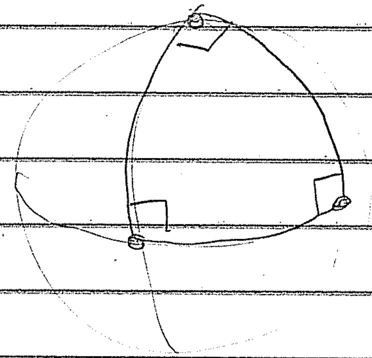




Because $\alpha + \beta + 90^\circ = 180^\circ$
(Angles in a triangle sum to 180°).

- Why is THAT true?

It's NOT true on a sphere (e.g. Earth)



three right angles!

$90^\circ + 90^\circ + 90^\circ \neq 180^\circ$

Oh well . . .

At least I showed that

(if $\theta = 90^\circ$ then area(C) = area(A) + area(B))

$\Uparrow$

angles in a triangle sum to 180°

$\Uparrow$

maybe you have more complaints.

- what is "area"?
- what is a "triangle"?

⋮

When can I stop?!

At some point we "just stop"

Pythagorean Theorem

$\Uparrow$

angles in $\triangle$ sum to $180°$

$\Uparrow$

$\vdots$

$\Uparrow$

$\boxed{\text{AXIOMS.}}$

Hopefully the axioms are "self-evident" ("need no proof")

If you still don't agree, that's your problem.

The idea of axioms goes back to the Greeks

Euclid, "The Elements", ~300 BC.
- 13 books
- Book 1 is a proof of the Pythagorean Theorem

Fri Aug 24, 2012

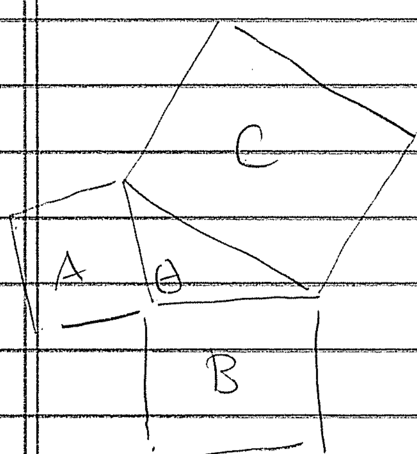
MTH 230
Intro to "Abstract Math"

Webpage:
www.math.miami.edu/~armstrong
(Wednesday's notes are posted)

Q: What is "Mathematics"?
(Submit your answer)

My Answer:
- math allows humans to agree on things.
- a PROOF is an attempt to persuade
- a THEOREM is a thing we've been persuaded to agree about.

Example: The Pythagorean Theorem
(the first theorem).

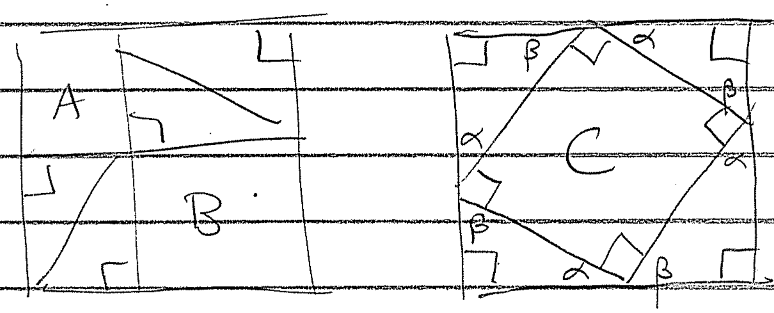


If $\theta = 90°$ then

area(A) + area(B)
= area(C).

Do you agree?

Proof: Assume that $\theta = 90°$ and consider two squares



The right figure is a square because
$\alpha + \beta + 90° = 180°$
(angles in a Δ sum to $180°$).
The squares have equal area.
Remove the 4 triangles from each.
They still have equal area.

Hence $\underbrace{\text{area}(A) + \text{area}(B)}_{\text{~~on the left~~}} = \underbrace{\text{area}(C)}_{\text{~~on the right~~}}$ $\boxtimes$

Now do you agree?
If yes, then we call it a "Theorem".

Pythagorean Theorem

$\Uparrow$

angles in Δ sum to 180°

$\Uparrow$

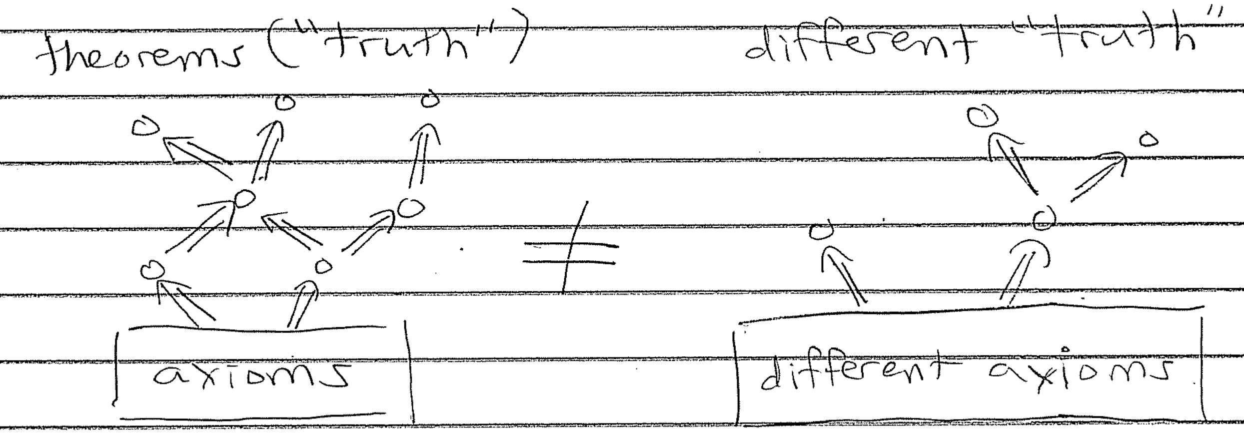
(Why?)

$\Uparrow$

$\boxed{\text{AXIOMS.}}$

We have to stop somewhere.
AXIOMS are the rules of the game.
They should be "self-evident"
or "so obvious that no one will disagree"
- choose your axioms carefully!

However:



Mathematical "truth" is relative.

Q: So what are the ~~best~~ axioms? I.O.U.
most popular

Q: What does $\Longrightarrow$ mean?

☆ Logical Principle ("material implication")

Truth flows along $\Longrightarrow$
(Equivalently, Falsity flo. backwards).

i.e.

$T \Rightarrow T$	$F \Rightarrow T$	$F \Rightarrow F$	$T \Rightarrow F$.
OK	OK	ok	BAD. it violates the principle.

(Note that everything is preserved if we switch T with F and $\Longrightarrow$ with $\Longleftarrow$ simultaneously.

This is called a "symmetry")

Application: The 2nd Oldest Theorem.

$\sqrt{2}$ is not a ratio of whole numbers.

Proof: Assume that $\sqrt{2}$ is a fraction. Then we can write $\sqrt{2} = a/b$ where a, b are whole numbers with no common denominator (say a/b is in "lowest terms").

Square both sides to get $2 = a^2/b^2$ and multiply by b^2 to get $a^2 = 2b^2$. Hence a^2 is an EVEN number. This implies (?) that a is EVEN, say $a = 2k$.

But then $4k^2 = a^2 = 2b^2$. Divide by 2 to get $b^2 = 2k^2$. Hence b^2 is EVEN, and so is b.

We see that a and b are both EVEN, hence a/b is NOT in lowest terms. CONTRADICTION. $\boxtimes$

What did we do ???

Picture: The proof with English removed

(1) "$\sqrt{2}$ is a fraction"

$\Downarrow$

"$\sqrt{2} = a/b$ for some a, b with no common divisor"

(2)

$\Downarrow$

...

$\Downarrow$

"a, b are both divisible by 2".

(3)

What could the truth values be (T or F)?

If (1) = T then (2) = (3) = T by implication. But (2) & (3) cannot simultaneously be T.
Having no other option, we conclude that (1) = F

///

Wed Aug 29

Business:

Exam 1 — Fri Sept 21
Exam 2 — Wed Oct 24
Exam 3 — Fri Nov 30

Office Hours:

Monday 1 – 2:30
Wed 3 – 4
and by appointment

Now: Informal Logic

Rule 1 ("excluded middle"):
Every mathematical statement is either T or F.

Rule 2 ("material implication"):
T flows along arrows $\Rightarrow$.

i.e. $T \Rightarrow T$ ✓ $F \Rightarrow T$ ✓ $F \Rightarrow F$ ✓ $T \Rightarrow F$ ✗

Application to $\sqrt{2}$:

First we need a lemma (little helper theorem).

Lemma: If n^2 is even then so is n.

TRY Proof: We must show that

$$\underbrace{\text{"}n^2 \text{ is even"}}_{P} \Rightarrow \underbrace{\text{"}n \text{ is even"}}_{Q}$$

To do this we must show that $(P,Q) = (T,F)$ doesn't happen. But if $Q = F$ then n is odd so $n = 2k+1$ for some k. In this case,

$$\begin{aligned} n^2 &= (2k+1)^2 = 4k^2 + 4k + 1 \\ &= 4(k^2+k) + 1 \\ &= 2(2k^2+2k) + 1 \\ &= 2(\text{something}) + 1 \end{aligned}$$

is odd. Hence $P = F$ ☒

We can be cleaner...

Principle ("the contrapositive").
Let P, Q be statements. The compound statements

$$\text{"}P \Rightarrow Q\text{"} \quad \text{and} \quad \text{"NOT } P \Leftarrow \text{NOT } Q\text{"}$$

are <u>logically</u> <u>equivalent</u>.

<u>Proof</u>: Rule 2 is symmetric under switching $T \leftrightarrow F$ and $\Rightarrow \leftrightarrow \Leftarrow$ $\square$

Example.

$$\text{"}n^2 \text{ even} \Rightarrow n \text{ even"} \qquad P \Rightarrow Q$$

$$\wr\wr$$

$$\text{"}n \text{ odd} \Rightarrow n^2 \text{ odd"} \qquad \text{NOT } Q \Rightarrow \text{NOT } P.$$

TRY AGAIN.

GOOD

Lemma: If n^2 is even then so is n.
Proof: We will show the contrapositive. So suppose that n is odd, say $n = 2k+1$. Then $n^2 = 2(k^2+k)+1$, hence n^2 is odd $\square$

Finally,

Theorem: $\sqrt{2}$ is not a fraction.

Proof: Suppose (for contradiction) that $\sqrt{2}$ is a fraction. Then we can write $\sqrt{2} = a/b$ with a, b COPRIME. Square to get $2 = a^2/b^2$ and multiply by b^2 to get $a^2 = 2b^2$. By the Lemma this implies a is EVEN, say $a = 2k$. But then $4k^2 = a^2 = 2b^2$. Dividing by 2 gives $b^2 = 2k^2$ and the Lemma implies that b is EVEN. But this contradicts the fact that a, b are COPRIME. Hence $\sqrt{2}$ is NOT a fraction. $\boxtimes$

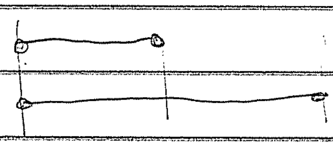
Homework: Prove that $\sqrt{3}$ is not a fraction.

Q: Who cares?

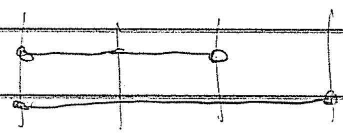
Math began as a religion

Pythagoras ~500BC

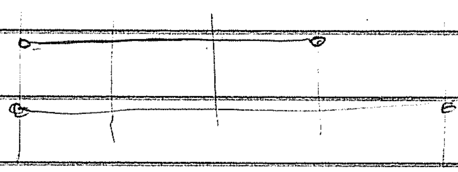
- "all is number"



1:2 = octave



2:3 = perfect fifth



3:4 = perfect fourth

etc.

- Theory: EVERYTHING can be explained in terms of ratios of (hopefully small) whole numbers.

"music of the spheres"

- don't eat beans

Fri Aug 31

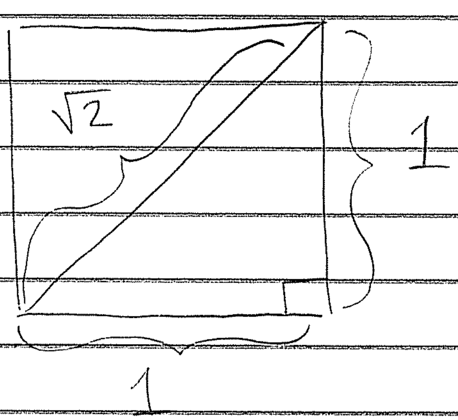
HW 1 due Fri Sept 7

The Pythagoreans ~ 500 BC

(① "don't eat beans")
② "all is number"
③ "number" = "ratio of whole numbers"
④ $\sqrt{2}$ is not a "number"

Uh-Oh ...

The discoverer was thrown overboard



Suppose

$\sqrt{2}$ = a "units"

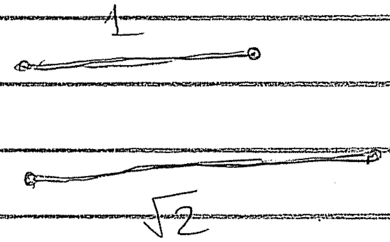
1 = b "units"

So ...

impossible!

$\sqrt{2}/a = 1$ "unit" $= 1/b \Rightarrow \sqrt{2} = a/b$

The side and diagonal have no common unit of measure. They are "incommensurable" i.e. these strings sound <u>bad</u> ("dissonant")



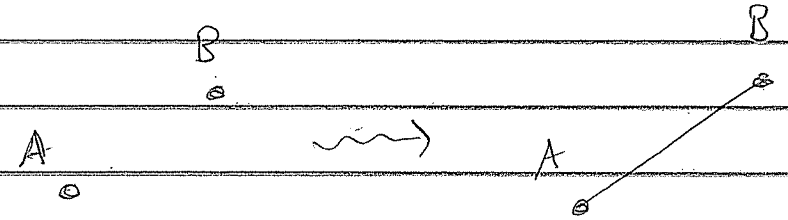
This "crisis of incommensurables" caused the Greeks to found their math on "length" instead of "number"
[Rift persisted until Descartes (1637)]

The First Axiomatic System:

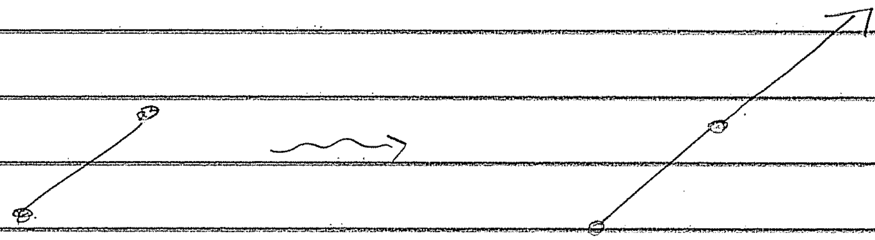
Euclid, "The Elements", ~300 BC
- <u>every</u> educated person read this until about 1900 (Abraham Lincoln)

10 Axioms
- 5 "postulates"
- 5 "common notions"

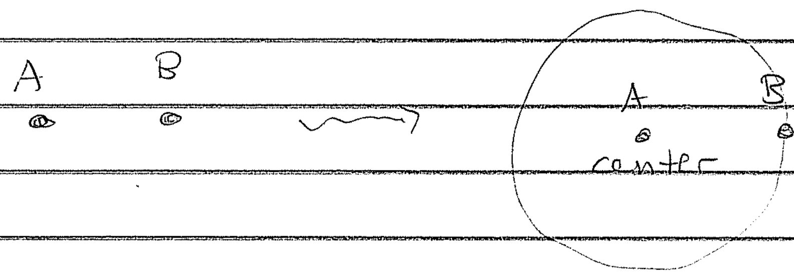
P1



P2

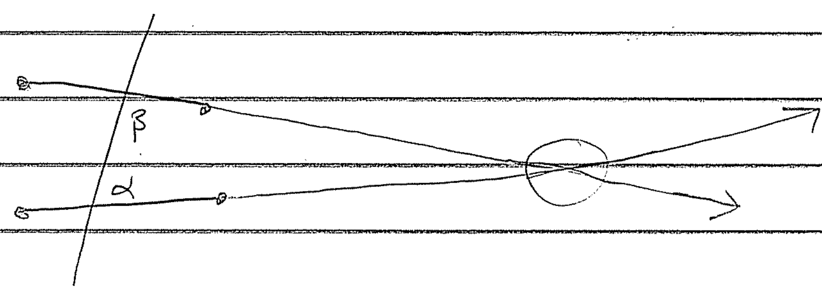


P3



P4 All right angles are "equal".
(necessary?)

P5



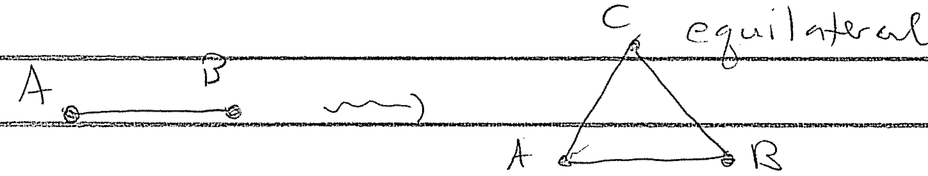
$\alpha + \beta < 180^\circ \Rightarrow$ the lines will meet

"The parallel postulate"

Common Notions CN1 – CN5 describe properties of "=" and "<".

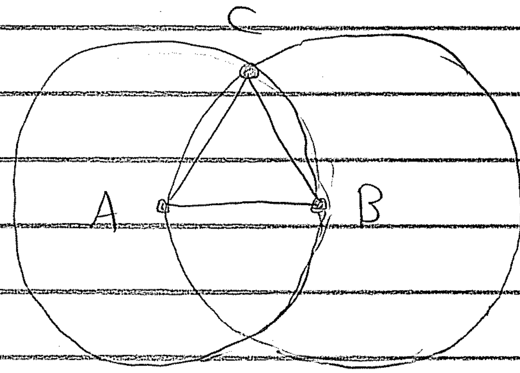
The Elements contains 13 = XIII books.
Book I as 48 propositions ("theorems")

Prop. I.1



"equilateral triangles exist"
(i.e. are constructible)

Proof:



Draw circle center A radius AB. (P3)
Draw circle center B radius AB (P3)
Let C be point of intersection ?
Draw triangle ABC (P1)

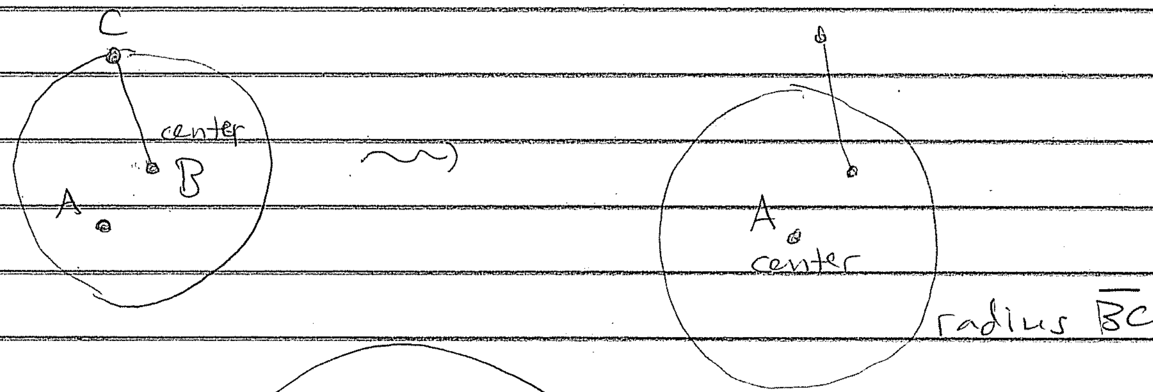
Have $\left.\begin{array}{l}\overline{AC} = \overline{AB} \\ \overline{BC} = \overline{AB}\end{array}\right\}$ definition of circle.

Hence also $\overline{AC} = \overline{BC}$ (CN 1)

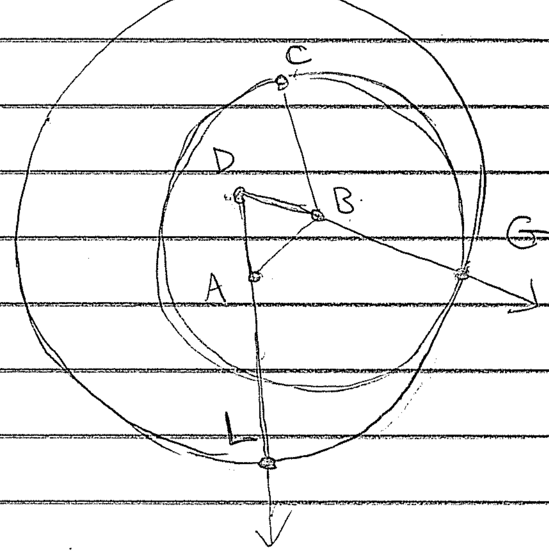
$\therefore$ $\triangle ABC$ is equilateral

Q.E.D.

Prop. I.2 : To move a circle.



Proof:



Draw equilateral $\triangle DAB$	I.1
Extend DB to G	P2
Draw circle center D radius DG	P3
Extend DA to L	P2
$\overline{DL} = \overline{DG}$ and $\overline{DA} = \overline{DB}$	Definition.
Hence $\overline{AL} = \overline{BG}$	CN3
$\overline{BC} = \overline{BG}$	Definition
Hence $\overline{AL} = \overline{BC}$	CN1
Finally, draw circle at A radius AL	P3

Q.E.D.

it continues . . . to 48 propositions
(why did Abraham Lincoln read this?)

Prop I.47 is the
Pythagorean Theorem.

Q: So, what is Prop I.48 ?!

~~(Postponed)~~
~~Discussion: Let P, Q be statements.~~
~~Then~~
~~"$P \Rightarrow Q$" = "NOT $Q \Rightarrow$ NOT P"~~
~~CONTRAPOSITIVE~~

~~But~~
~~"$P \Rightarrow Q$" $\neq$ "$Q \Rightarrow P$"~~
~~CONVERSE~~

~~e.g. let P = "$x > 0$"~~
~~Q = "$x^2 > 0$"~~

~~Then "$P \Rightarrow Q$" = T~~
~~and "NOT $Q \Rightarrow$ NOT P"~~
~~= "$x^2 \leq 0 \Rightarrow x \leq 0$" = T~~

~~BUT "$Q \Rightarrow P$"~~
~~= "$x^2 > 0 \Rightarrow x > 0$" = F~~

Wed, Sept 5

HW 1 due Friday.

Office Hours Today: 3:45 → 4:45.
(special time)

Euclid & "What is Math?" Handouts

Euclid, "The Elements" ~300 BC.
XIII books.

Book I = proof of Pythagorean Theorem
⋮
Book XIII classifies the 5 platonic solids.

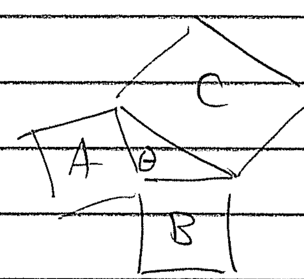
tetrahedron	fire
cube	earth
octahedron	air
icosahedron	water
dodecahedron	"aether"/cosmic sphere.

Book I has 48 Propositions

Prop I.47

If $\theta = 90°$ then

area(A) + area(B) = area(C)



Q: So What is Prop I.48 ?!

Discussion: Let P, Q be statements. Then

"$P \Rightarrow Q$" = "NOT Q $\Rightarrow$ NOT P"

the contrapositive

But

"$P \Rightarrow Q$" $\neq$ "$Q \Rightarrow P$"

the converse

not necessarily equal (they may be)

e.g. let P = "$x > 0$"

Q = "$x^2 > 0$"

Then "$P \Rightarrow Q$" = "if $x > 0$ then $x^2 > 0$" = T

and "NOT Q $\Rightarrow$ NOT P" = "if $x^2 \leq 0$ then $x \leq 0$" = T

But "$Q \Rightarrow P$" = "if $x^2 > 0$ then $x > 0$" = F

counterexample: take $x = -2 < 0$

$x^2 = 4 > 0$

///

It may happen that $P \Rightarrow Q$ and $Q \Rightarrow P$ are both true. In this case we write

$P \Longleftrightarrow Q$

"P if and only if Q".

("$P \Rightarrow Q$" P only if Q

"$Q \Rightarrow P$" P if Q.)

Logically:

"$P \Leftrightarrow Q$" = "$P \Rightarrow Q$ AND $Q \Rightarrow P$"

To prove it you must prove $P \Rightarrow Q$ and $Q \Rightarrow P$ SEPARATELY.

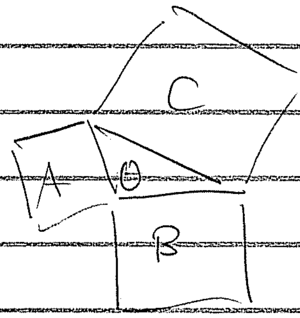
eg. $\underbrace{\text{"n is an integer"}}$

Claim: For $n \in \mathbb{Z}$ we have n^2 even if and only if n is even.

Proof: We showed n^2 even $\Rightarrow$ n even previously.

Now we will show n even $\Rightarrow$ n^2 even. So suppose $n = 2k$ for some $k \in \mathbb{Z}$. Then $n^2 = 4k^2 = 2(2k^2)$ is even. $\blacksquare$

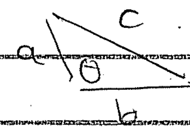
Euclid's Prop I.48: The converse of the Pythagorean Theorem



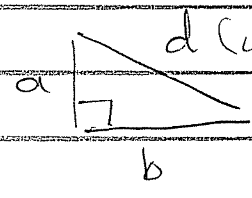
<u>If</u> area(A) + area(B) = area(C)

<u>then</u> $\theta = 90°$.

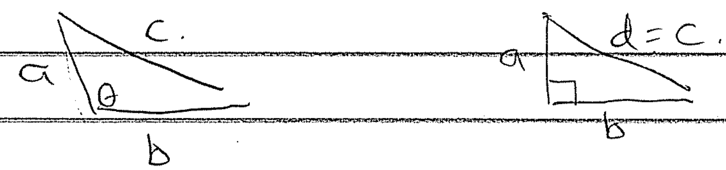
Proof (Modern Language): Label the side lengths of the <u>given</u> triangle.



We assume that $a^2 + b^2 = c^2$. We want to show that $\theta = 90°$. To do this we draw a (new) right triangle with side lengths a, b, d, where d is unknown to us. (Euclid I.2 and I.11 allow this.)



Prop. I.47 (Pyth. Thm.) implies that $a^2 + b^2 = d^2$. But we assumed $a^2 + b^2 = c^2$. Hence $d^2 = c^2$ and $d = c$.

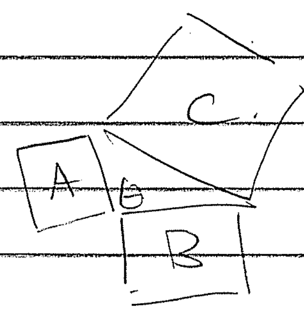


The two triangles have the same side lengths, hence (Euclid I.8) they also have the same angles.

We conclude that $\theta = 90^\circ$ ∎

Putting Props I.47 & I.48 together we have

Theorem:



$$\theta = 90^\circ \iff \text{area}(A) + \text{area}(B) = \text{area}(C).$$

(Application in ancient engineering?)