Math 224 **Summer 2019**
Homework 4 **Drew Armstrong**

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1. Each box of a certain brand of cereal comes with a toy inside. If there are n possible toys and if the toys are distributed uniformly, how many boxes of cereal do you expect to buy until you get all of the toys?

(a) Assume that you already have ℓ of the toys and let X_ℓ be the number of boxes until you get a new toy. Compute the expected value $E[X_\ell]$. [Hint: Think of each box of cereal as a coin flip with $H =$"new toy" and $T =$"old toy." Thus X_ℓ is a geometric random variable. What is $P(H)$ in this case?]

(b) Let X be the number of boxes you purchase until you get all n toys. Thus we have

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Use part (a) and linearity to compute the expected value $E[X]$.

(c) Application: Suppose you continue to roll a fair 6-sided die until you see all six sides. How many rolls do you expect to make?

(a) Fix $\ell \in \{0, 1, \ldots, n-1\}$ and assume that you already have ℓ of the toys. Until you get a new toy, each box of cereal can temporarily be thought of as a coin flip with $H =$ "a new toy" and $T =$ "an old toy." Since the n toys are distributed randomly this means that

$$P(H) = p = \frac{n-\ell}{n} \qquad \text{and} \qquad P(T) = q = \frac{\ell}{n}.$$

If X_ℓ is the "number of boxes until a new toy" (i.e., the "number of flips until heads") then we observe that X_ℓ is a geometric random variable. Hence we expect to get a new toy after purchasing $1/p$ boxes:

$$E[X_\ell] = \frac{1}{p} = \frac{n}{n-\ell}.$$

(b) If X is the total number of boxes we need to purchase until getting all n toys then we can express this as a sum of the random variables from part (a):

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Now we can apply linearity to compute the expected value:

$$\begin{aligned} E[X] &= E[X_0] + E[X_1] + E[X_2] + \cdots + E[X_{n-1}] \\ &= \frac{n}{n-0} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}. \end{aligned}$$

Sadly, this formula does not simplify.

(c) Application: Start rolling a fair 6-sided die and let X be the number of rolls until you see all six faces. Then according to part (b) we have

$$\begin{aligned} E[X] &= \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} \\ &= 1 + 1.2 + 1.5 + 2 + 3 + 6 = 14.7. \end{aligned}$$

In other words, we see our 1st new face on the first roll. Then it takes 1.2 rolls to see the 2nd face and 1.5 rolls to see the 3rd face, etc. After we have seen 5 faces it will take on average 6 rolls until we see the final face. In total, we expect to roll the die 14.7 times.

[Moral of the Story: Linearity of expectation is useful.]

2. Let X be a random variable satisfying

$$E[X] = 2 \qquad \text{and} \qquad E[X^2] = 7.$$

Use this information to compute the following:

$$\text{Var}(X), \quad E[X+3], \quad E[(X+3)^2] \quad \text{and} \quad \text{Var}(X+3).$$

First we compute the variance:

$$\text{Var}(X) = E[X^2] - E[X]^2 = 7 - (2)^2 = 3.$$

Then we use the linearity of expectation to compute

$$E[X+3] = E[X] + E[3] = E[X] + 3 = 2 + 3 = 5$$

and

$$\begin{aligned} E[(X+3)^2] &= E[X^2 + 6X + 9] \\ &= E[X^2] + 6E[X] + 9 \\ &= 7 + 6(2) + 9 \\ &= 28. \end{aligned}$$

Finally, there are two ways to compute the variance of $X + 3$. Directly:

$$\text{Var}(X+3) = E[(X+3)^2] - E[X+3]^2 = 28 - (5)^2 = 3.$$

Or we could just use the fact that

$$\text{Var}(X+3) = \text{Var}(X) = 3.$$

3. Let X be a random variable with mean $E[X] = \mu$ and variance $\text{Var}(X) = \sigma^2 \neq 0$. Compute the mean and variance of the random variable Y defined by

$$Y = \frac{X - \mu}{\sigma}.$$

This is pure algebra. For the expected value we have

$$E[Y] = E\left[\frac{1}{\sigma}(X-\mu)\right] = \frac{1}{\sigma}E[X-\mu] = \frac{1}{\sigma}\left(E[x] - \mu\right) = \frac{1}{\sigma}(\mu - \mu) = 0.$$

Then for the variance we have

$$\text{Var}(Y) = \text{Var}\left(\frac{1}{\sigma}(X-\mu)\right) = \left(\frac{1}{\sigma}\right)^2 \text{Var}(X-\mu) = \frac{1}{\sigma^2}\text{Var}(X) = \frac{1}{\sigma} \cdot \sigma^2 = 1.$$

4. Let U be the uniform random variable with pdf (probability density function) defined by

$$f_U(x) := \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Compute the first two moments $\mu = E[U]$ and $E[U^2]$.
(2) Compute the variance $\sigma^2 = \text{Var}(U) = E[U^2] - E[U]^2$ and standard deviation σ.
(3) Compute the probability $P(\mu - \sigma \leq U \leq \mu + \sigma)$.
(4) Draw the graph of f_U, showing the interval $\mu \pm \sigma$ in your picture.

(a) By definition we have

$$E[U] = \int_0^1 t \cdot 1\, dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$

and

$$E[U^2] = \int_0^1 t^2 \cdot 1\, dt = \left.\frac{t^3}{3}\right|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

(b) Then the variance is

$$\sigma^2 = \text{Var}(U) = E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

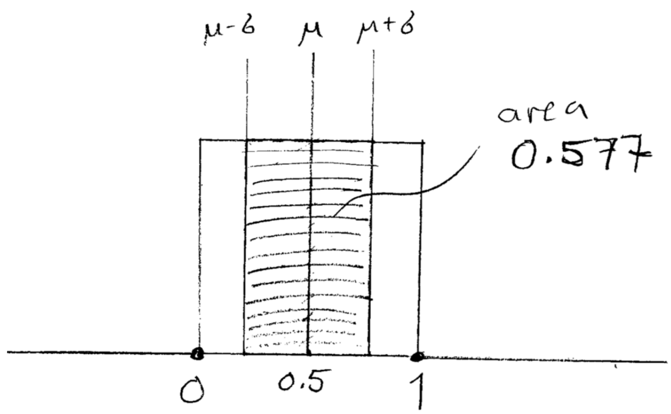
and the standard deviation is

$$\sigma = \sqrt{\text{Var}(U)} = \sqrt{1/12} = 0.289.$$

(c) The probability that U within one standard deviation of the mean is

$$\begin{aligned} P(\mu - \sigma < U < \mu + \sigma) &= P(0.211 < U < 0.789) \\ &= \int_{0.211}^{0.789} 1\, dt = t\Big|_{0.211}^{0.789} = (0.789) - (0.211) = 0.577 = 57.7\%. \end{aligned}$$

(d) We can also view this as the area of a rectangle with base $2\sigma = 0.577$ and height 1:



5. Let X be a continuous random variable with pdf defined as follows:

$$f_X(x) = \begin{cases} c \cdot x^2 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the value of the constant c. [Hint: The total probability is 1.]
(b) Find the mean $\mu = E[X]$ and standard deviation $\sigma = \sqrt{\text{Var}(X)}$.
(c) Compute the probability $P(\mu - \sigma \leq X \leq \mu + \sigma)$.
(d) Draw the graph of f_X, showing the interval $\mu \pm \sigma$ in your picture.

(a) To find c we use the fact that the total area under the pdf equals 1. Thus we have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(x)\, dx \\ &= \int_0^1 c \cdot x^2\, dx \\ &= c \cdot \left.\frac{x^3}{3}\right|_0^1 = \frac{c}{3}, \end{aligned}$$

and it follows that $c = 3$.

(b) By definition, the first moment is

$$\begin{aligned} \mu = E[X] &= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx \\ &= \int_0^1 x \cdot 3x^2\, dx \\ &= 3 \cdot \left.\frac{x^4}{4}\right|_0^1 = \frac{3}{4}. \end{aligned}$$

Then the second moment is

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx \\ &= \int_0^1 x^2 \cdot 3x^2\, dx \\ &= 3 \cdot \left.\frac{x^5}{5}\right|_0^1 = \frac{3}{5}, \end{aligned}$$

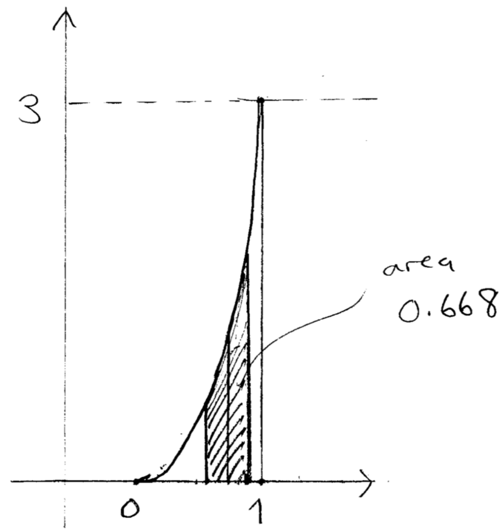
and hence

$$\begin{aligned} \sigma^2 = \text{Var}(X) &= E[X^2] - E[X]^2 = (3/5) - (3/4)^2 = 3/80, \\ \sigma &= \sqrt{3/80} = 0.1936. \end{aligned}$$

(c) We have

$$\begin{aligned} P(\mu - \sigma \le X \le \mu + \sigma) &= P(0.5564 \le X \le 0.9436) \\ &= \int_{0.5564}^{0.9436} 3x^2\, dx \\ &= 3 \cdot \left.\frac{x^3}{3}\right|_{0.5564}^{0.9436} = (0.9436)^3 - (0.5564)^3 = 66.80\%. \end{aligned}$$

(d) Here is a picture of the pdf:



6. Let Z be the standard normal random variable, with pdf defined as follows:

$$f_Z(x) = n(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}.$$

Let $\Phi(z)$ be the associated cdf (cumulative density function), which is defined by

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} n(x)\, dx.$$

Use the attached table to compute the following probabilities:

(a) $P(0 < Z < 0.5)$,
(b) $P(Z < -0.5)$,
(c) $P(Z > 1),\ P(Z > 2),\ P(Z > 3)$.
(d) $P(|Z| < 1),\ P(|Z| < 2),\ P(|Z| < 3)$,

(a) $P(0 < Z < 0.5) = \Phi(0.5) - \Phi(0) = 0.6915 - 0.5000 = 19.15\%$.

(b) $P(Z < -0.5) = \Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 30.85\%$.

(c)

$$P(Z > 1) = 1 - P(Z < 1) = 1 - \Phi(1) = 1 - 0.8413 = 15.87\%,$$
$$P(Z > 2) = 1 - P(Z < 2) = 1 - \Phi(2) = 1 - 0.9772 = 2.28\%,$$
$$P(Z > 3) = 1 - P(Z < 3) = 1 - \Phi(3) = 1 - 0.9987 = 0.13\%.$$

(d) For any positive c we have

$$P(|Z| < c) = P(-c < Z < c) = \Phi(c) - \Phi(-c) = \Phi(c) - [1 - \Phi(c)] = 2 \cdot \Phi(c) - 1.$$

Thus we have

$$P(|Z| < 1) = 2 \cdot \Phi(1) - 1 = 2 \cdot 0.8413 - 1 = 68.26\%,$$
$$P(|Z| < 2) = 2 \cdot \Phi(2) - 1 = 2 \cdot 0.9772 - 1 = 95.44\%,$$
$$P(|Z| < 3) = 2 \cdot \Phi(3) - 1 = 2 \cdot 0.9987 - 1 = 99.74\%.$$