Math 224 **Summer 2019**
Homework 4 **Drew Armstrong**

---

1. Each box of a certain brand of cereal comes with a toy inside. If there are n possible toys and if the toys are distributed uniformly, how many boxes of cereal do you expect to buy until you get all of the toys?

(a) Assume that you already have ℓ of the toys and let X_ℓ be the number of boxes until you get a new toy. Compute the expected value $E[X_\ell]$. [Hint: Think of each box of cereal as a coin flip with $H =$"new toy" and $T =$"old toy." Thus X_ℓ is a geometric random variable. What is $P(H)$ in this case?]

(b) Let X be the number of boxes you purchase until you get all n toys. Thus we have

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Use part (a) and linearity to compute the expected value $E[X]$.

(c) Application: Suppose you continue to roll a fair 6-sided die until you see all six sides. How many rolls do you expect to make?

2. Let X be a random variable satisfying

$$E[X] = 2 \quad \text{and} \quad E[X^2] = 7.$$

Use this information to compute the following:

$$\text{Var}(X), \quad E[X+3], \quad E[(X+3)^2] \quad \text{and} \quad \text{Var}(X+3).$$

3. Let X be a random variable with mean $E[X] = \mu$ and variance $\text{Var}(X) = \sigma^2 \neq 0$. Compute the mean and variance of the random variable Y defined by

$$Y = \frac{X - \mu}{\sigma}.$$

4. Let U be the uniform random variable with pdf (probability density function) defined by

$$f_U(x) := \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) Compute the first two moments $\mu = E[U]$ and $E[U^2]$.
(2) Compute the variance $\sigma^2 = \text{Var}(U) = E[U^2] - E[U]^2$ and standard deviation σ.
(3) Compute the probability $P(\mu - \sigma \leq U \leq \mu + \sigma)$.
(4) Draw the graph of f_U, showing the interval $\mu \pm \sigma$ in your picture.

5. Let X be a continuous random variable with pdf defined as follows:

$$f_X(x) = \begin{cases} c \cdot x^2 & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the value of the constant c. [Hint: The total probability is 1.]
(b) Find the mean $\mu = E[X]$ and standard deviation $\sigma = \sqrt{\text{Var}(X)}$.
(c) Compute the probability $P(\mu - \sigma \leq X \leq \mu + \sigma)$.
(d) Draw the graph of f_X, showing the interval $\mu \pm \sigma$ in your picture.

6. Let Z be the standard normal random variable, with pdf defined as follows:

$$f_Z(x) = n(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}.$$

Let $\Phi(z)$ be the associated cdf (cumulative density function), which is defined by

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} n(x)\, dx.$$

Use the attached table to compute the following probabilities:

(a) $P(0 < Z < 0.5)$,
(b) $P(Z < -0.5)$,
(c) $P(Z > 1),\ P(Z > 2),\ P(Z > 3)$.
(d) $P(|Z| < 1),\ P(|Z| < 2),\ P(|Z| < 3)$,

Standard Normal Probabilities

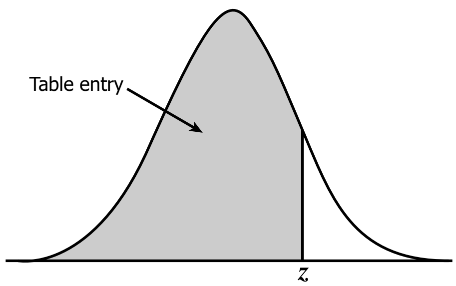

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998