Math 224 **Summer 2019**
Homework 2 **Drew Armstrong**

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1. Suppose that a fair s-sided die is rolled n times.

(a) If the i-th side is labeled a_i then we can think of the sample space S as the set of all words of length n from the alphabet $\{a_1, \ldots, a_s\}$. Find $\#S$.

(b) Let E be the event that "the 1st side shows up k_1 times, and ... and the s-th side shows up k_s times. Find $\#E$. [Hint: The elements of E are words of length n in which the letter a_i appears k_i times.]

(c) Compute the probability $P(E)$. [Hint: Since the die is fair you can assume that the outcomes in S are equally likely.]

The purpose of this problem was to encourage you to look at the course notes.

(a) To form a word of length n from the alphabet $\{a_1, \ldots, a_s\}$, there are s choices for the 1st letter, then s choices for the 2nd letter, then ... then s choices for the nth letter. Hence

$$\#S = \#(\text{words}) = \underbrace{s}_{\text{1st letter}} \times \underbrace{s}_{\text{2nd letter}} \times \cdots \times \underbrace{s}_{n\text{th letter}} = s^n.$$

(b) Note that $k_1 + k_2 + \cdots + k_s = n$. To form a word from k_1 copies of a_1, k_2 copies of a_2, ... and k_s copies of a_s, we can first form a permutation of the n symbols in $n!$ ways, but then we have to divide by $k_i!$ for each i to remove the orderings between the indistinguishable copies of a_i. Hence

$$\#E = \binom{n}{k_1, k_2, \ldots, k_s} = \frac{n!}{k_1!k_2!\cdots k_s!}.$$

(c) Since all outcomes are equally likely we have

$$P(E) = \frac{\#E}{\#S} = \binom{n}{k_1, k_2, \ldots, k_s} \cdot \frac{1}{s^n}.$$

2. Suppose that a fair six-sided die has 1 side painted red, 2 sides painted green and 3 sides painted blue. Suppose that you roll the die $n = 4$ times and let R, G, B be the number of times you see red, green and blue, respectively.

(a) Compute $P(R = 1, G = 1, B = 2)$.

(b) Compute $P(R = 1, G = 2, B = 1)$.

(c) Compute $P(R \geq 1)$. [Hint: Treat the die as a coin.]

(d) Compute $P(G = B)$. [Hint: What are the possible values of R, G, B?]

(a) Since dice rolls are independent, we use the formula for multinomial probability:

$$P(R = 1, G = 1, B = 2) = \frac{4!}{1!1!2!}\left(\frac{1}{6}\right)^1\left(\frac{2}{6}\right)^1\left(\frac{3}{6}\right)^2 = \frac{1}{6}.$$

(b) Same formula:

$$P(R = 1, G = 2, B = 1) = \frac{4!}{1!2!1!}\left(\frac{1}{6}\right)^1\left(\frac{2}{6}\right)^2\left(\frac{3}{6}\right)^1 = \frac{1}{9}.$$

(c) This time we treat the die as a coin with "heads"="red" and "tails"="not red." Since $P(H) = 1/6$ and $P(T) = 5/6$ we have

$$P(\text{at least one red}) = 1 - P(\text{no red}) = 1 - P(T)^4 = 1 - \left(\frac{5}{6}\right)^4 = 51.8\%.$$

(d) This time there is no short cut. Since $R+G+B = 4$ we find that there are three different ways to get $G = B$. To compute the probability we add them up:

$$\begin{aligned} P(G = B) &= P(R = 0, G = 2, B = 2) + P(R = 2, G = 1, B = 1) + P(R = 4, G = 0, B = 0) \\ &= 1/6 + 1/18 + 1/1296 \\ &= 22.3\%. \end{aligned}$$

3. In a certain state lottery four numbers are drawn (one and at a time and with replacement) from the set $\{1, 2, 3, 4, 5, 6\}$. You win if any permutation of your selected numbers is drawn. Rank the following selections in order of how likely each is to win.

(a) You select $1, 2, 3, 4$.
(b) You select $1, 2, 3, 3$.
(c) You select $1, 1, 2, 2$.
(d) You select $1, 2, 2, 2$.
(e) You select $1, 1, 1, 1$.

One way to solve this is to view it as an application of Problem 1. The problem is equivalent to rolling a fair $s = 6$-die $n = 4$ times, where the sides are labeled $\{1, 2, 3, 4, 5, 6\}$. The sample space has size

$$\#S = 6^4 = 1296.$$

In general, the number of outcomes in which 1 appears k_1 times, 2 appears k_2 times ... and 6 appears k_6 times is given by

$$\binom{4}{k_1, k_2, k_3, k_4, k_5, k_6} = \frac{4!}{k_1!k_2!k_3!k_4!k_5!k_6!}.$$

(a) The number of permutations of $1, 2, 3, 4$ is

$$\binom{4}{1, 1, 1, 1, 0, 0} = \frac{4!}{1!1!1!1!0!0!} = 24,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{24}{1296} = 1.85\%.$$

(b) The number of permutations of $1, 2, 3, 3$ is

$$\binom{4}{1, 1, 2, 0, 0, 0} = \frac{4!}{1!1!2!0!0!0!} = 12,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{12}{1296} = 0.93\%.$$

(c) The number of permutations of $1, 1, 2, 2$ is

$$\binom{4}{2, 2, 0, 0, 0, 0} = \frac{4!}{2!2!0!0!0!0!} = 6,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{6}{1296} = 0.46\%.$$

(d) The number of permutations of $1, 2, 2, 2$ is

$$\binom{4}{1, 3, 0, 0, 0, 0} = \frac{4!}{1!3!0!0!0!0!} = 4,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{4}{1296} = 0.31\%.$$

(e) The number of permutations of $1, 1, 1, 1$ is

$$\binom{4}{4, 0, 0, 0, 0, 0} = \frac{4!}{4!0!0!0!0!0!} = 1,$$

hence the probability of winning is

$$P(\text{winning}) = \frac{1}{1296} = 0.077\%.$$

Hence the order of winning is

$$(\text{a}) > (\text{b}) > (\text{c}) > (\text{d}) > (\text{e}).$$

4. A bridge hand consists of 13 (unordered) cards taken (at random and without replacement) from a standard deck of 52. Recall that a standard deck contains 13 hearts and 13 diamonds (which are red cards), 13 clubs and 13 spades (which ard black cards). Find the probabilities of the following hands.

(a) 2 hearts, 3 diamonds, 4 spades and 4 clubs.
(b) 2 hearts, 3 diamonds and 8 black cards.
(c) 5 red cards and 8 black cards.

Let S be the set of possible bridge hands. The number of ways to choose a bridge hand (13 unordered cards without replacement from a deck of 52) is approximately 635 billion:

$$\#S = \binom{52}{13} = \frac{52!}{13!49!} = 635,013,559,600.$$

(a) The number of ways to choose 2 hearts, 3 diamonds, 4 spades and 4 clubs is

$$\underbrace{\binom{13}{2}}_{\substack{\text{choose}\\\text{hearts}}} \times \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\\text{diamonds}}} \times \underbrace{\binom{13}{4}}_{\substack{\text{choose}\\\text{spades}}} \times \underbrace{\binom{13}{4}}_{\substack{\text{choose}\\\text{clubs}}} = 11,404,407,300,$$

hence the probability is

$$P(2\heartsuit\text{'s}, 3\diamondsuit\text{'s}, 4\spadesuit\text{'s}, 4\clubsuit\text{'s}) = \frac{\binom{13}{2}\binom{13}{3}\binom{13}{4}\binom{13}{4}}{\binom{52}{13}} = 1.8\%.$$

(a) The number of ways to choose 2 hearts, 3 diamonds and 8 black cards

$$\underbrace{\binom{13}{2}}_{\substack{\text{choose}\\\text{hearts}}} \times \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\\text{diamonds}}} \times \underbrace{\binom{26}{8}}_{\substack{\text{choose}\\\text{black cards}}} = 34,851,230,700,$$

hence the probability is

$$P(2\heartsuit\text{'s}, 3\diamondsuit\text{'s}, 8 \text{ black}) = \frac{\binom{13}{2}\binom{13}{3}\binom{26}{8}}{\binom{52}{13}} = 5.5\%.$$

(a) The number of ways to choose 5 red cards and 8 black cards is

$$\underbrace{\binom{26}{5}}_{\substack{\text{choose}\\\text{red cards}}} \times \underbrace{\binom{26}{8}}_{\substack{\text{choose}\\\text{black cards}}} = 102,766,449,500,$$

hence the probability is

$$P(5 \text{ red}, 8 \text{ black}) = \frac{\binom{26}{5}\binom{26}{8}}{\binom{52}{13}} = 16.2\%.$$

5. Let $E, F \subseteq S$ be two events in a sample space and define the *conditional probability*:

$$P(E|F) = P(E \cap F)/P(F).$$

We interpret the number $P(E|F)$ as "the probability that E happens, assuming that F happens." Now suppose that two cards are drawn (in order and without replacement) from a standard deck of 52 and consider the events

$$A = \{\text{the first card is a diamond}\}$$
$$B = \{\text{the second card is red}\}.$$

In this case, compute the following probabilities:

$$P(A), \quad P(B), \quad P(B|A), \quad P(A \cap B), \quad P(A|B).$$

Solution: Since $13/52 = 1/4$ of the cards are diamonds we have $P(A) = 1/4$. Similarly, since $26/52 = 1/2$ of the cards are red we have $P(B) = 1/2$.[1] If the first card is chosen to be a diamond then 25 of the remaining 51 card are red, hence

$$P(B|A) = \frac{25}{51}.$$

From this we can compute

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{1}{4} \cdot \frac{25}{51} = \frac{25}{204}.$$

Finally, applying the formula for $P(A|B)$ gives

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)} = \left(\frac{1}{4} \cdot \frac{25}{51}\right) / \left(\frac{1}{2}\right) = \frac{25}{102}.$$

[1] To see this we just ignore the first card. There are harder ways to compute $P(B)$ but you will always get the same answer.

Let's say we draw two cards, but we don't look at the first card. If the second card is red, then the previous calculation tells us that the unknown first card has a $25/102$ chance of being a diamond, i.e., slightly less than $1/4$. Does that make any sense?

6. Consider a classroom containing n students. Suppose we record each student's birthday as a number between 1 and 365 (we ignore leap years). Let S be the sample space.

(a) Explain why $\#S = 365^n$.
(b) Let E be the event that {no two students have the same birthday}. Compute $\#E$.
(c) Assuming that all birthdays are equally likely, compute the probability of the event

$$E' = \{\text{at least two students have the same birthday}\}.$$

(d) Find the smallest value of n such that $P(E') > 50\%$.

(a) Suppose that the students are ordered (say, alphabetically by last name). Then we have

$$\#S = \underbrace{365}_{\substack{\text{1st student's}\\ \text{birthday}}} \times \underbrace{365}_{\substack{\text{2nd student's}\\ \text{birthday}}} \times \cdots \times \underbrace{365}_{\substack{n\text{th student's}\\ \text{birthday}}} = 365^n.$$

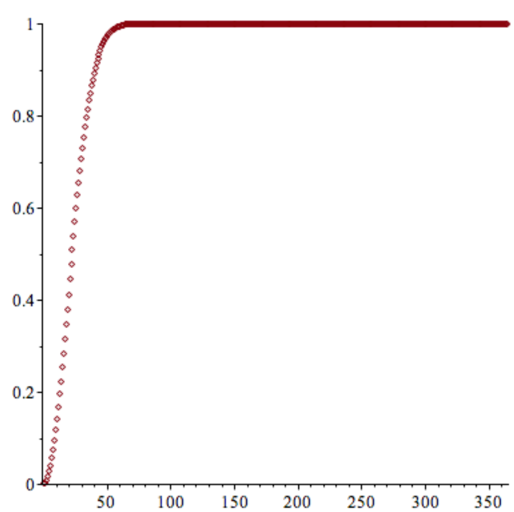
(b) If no two students are allowed to have the same birthday then for $n \geq 366$ we have $\#E = 0$ and for $n \leq 365$ we have

$$\#E = \underbrace{365}_{\substack{\text{1st student's}\\ \text{birthday}}} \times \underbrace{364}_{\substack{\text{2nd student's}\\ \text{birthday}}} \times \cdots \times \underbrace{(365 - n + 1)}_{\substack{n\text{th student's}\\ \text{birthday}}} = \frac{365!}{(365-n)!}.$$

(c) If $n \geq 366$ we have $P(E) = 0$ and hence $P(E') = 1 - P(E) = 1$. If $n \leq 365$ then assuming all birthdays are equally likely gives

$$\begin{aligned} P(\text{at least two share a birthday}) &= 1 - P(\text{no two share a birthday}) \\ P(E') &= 1 - P(E) \\ &= 1 - \frac{\#E}{\#S} \\ &= 1 - \frac{365!/(365-n)!}{365^n}. \end{aligned}$$

(d) Here is a plot of the probabilites $P(E')$ for values of n from 1 to 365. Note that the probability rises from 0% when $n = 1$ to 100% when $n = 366$.



At some point the probability must cross 50% and it seems from the diagram that this happens around $n = 25$. To be precise, I used my computer to find the following:

- For $n = 22$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-22)!}{365^{22}} = 47.57\%.$$
- For $n = 23$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-23)!}{365^{23}} = 50.73\%.$$

Do you find the number 23 surprisingly small? That's why this problem is sometimes also called the **birthday paradox**.[2]

7. It was not easy to find a formula for the entries of Pascal's Triangle. However, once we've found the formula it is not difficult to check that the formula is correct.

(a) Explain why $n! = n \cdot (n-1)!$.

(b) Use part (a) to verify that
$$\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{n!}{k!(n-k)!}.$$
[Hint: Try to get a common denominator.]

Part (a) is self-explanatory. For part (b) we use part (a) to get a common denominator:
$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \frac{(n-1)!}{k!\,[(n-1)-k]!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \left(\frac{k}{k}\right) \cdot \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \cdot \left(\frac{n-k}{n-k}\right) \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}
\end{aligned}$$
Then we use part (a) to simplify the numerator:
$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{[\cancel{k} + (n-\cancel{k})] \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}.
\end{aligned}$$

[2] In our class of $n = 11$ students there is a 14.1% chance that two students share a birthday.