Math 224 — Summer 2019
Homework 1 — Drew Armstrong

1. Suppose that a fair coin is flipped 6 times in sequence and let X be the number of "heads" that show up. Draw Pascal's triangle down to the sixth row (recall that the zeroth row consists of a single 1) and use your table to compute the probabilities $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5, 6$.

Here is Pascal's Triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Then since $2^6 = 64$ we have the following table of probabilities:

k	0	1	2	3	4	5	6
$P(X = k)$	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

2. Suppose that a fair coin is flipped 4 times in sequence.

(a) List all 16 outcomes in the sample space S.
(b) List the outcomes in each of the following events:
$A = \{\text{at most 3 heads}\}$,
$B = \{\text{more than 2 heads}\}$,
$C = \{\text{heads on the 3rd flip}\}$,
$D = \{\text{exactly 2 tails}\}$.
(c) Assuming that all outcomes are **equally likely**, use the formula $P(E) = \#E/\#S$ to compute the following probabilities:

$$P(A \cup B), \quad P(A \cap B), \quad P(C), \quad P(D), \quad P(C \cap D).$$

(a) The sample space is

$$
\begin{aligned}
S = \{&HHHH, \\
&HHHT, HHTH, HTHH, THHH, \\
&HHTT, HTHT, HTTH, THHT, THTH, TTHH, \\
&HTTT, THTT, TTHT, TTTH, \\
&TTTT\}
\end{aligned}
$$

(b) The events are

$$\begin{aligned}
A =&\{HHHT, HHTH, HTHH, THHH,\\
&HHTT, HTHT, HTTH, THHT, THTH, TTHH,\\
&HTTT, THTT, TTHT, TTTH,\\
&TTTT\},\\
B =&\{HHHH,\\
&HHHT, HHTH, HTHH, THHH\},\\
C =&\{HHHH,\\
&HHHT, HTHH, THHH,\\
&HTHT, THHT, TTHH,\\
&TTHT\},\\
D =&\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}.
\end{aligned}$$

(c) First observe that $A \cup B = S$ and hence $P(A \cup B) = P(S) = 1$. Next observe that

$$A \cap B = \{\text{exactly 3 heads}\} = \{HHHT, HHTH, HTHH, THHH\},$$

so that $P(A \cap B) = \#(A \cap B)/\#S = 4/16$. Then since $\#C = 8$ and $\#D = 6$ we have

$$P(C) = \frac{\#C}{\#S} = \frac{8}{16} \qquad \text{and} \qquad P(D) = \frac{\#D}{\#S} = \frac{6}{16}.$$

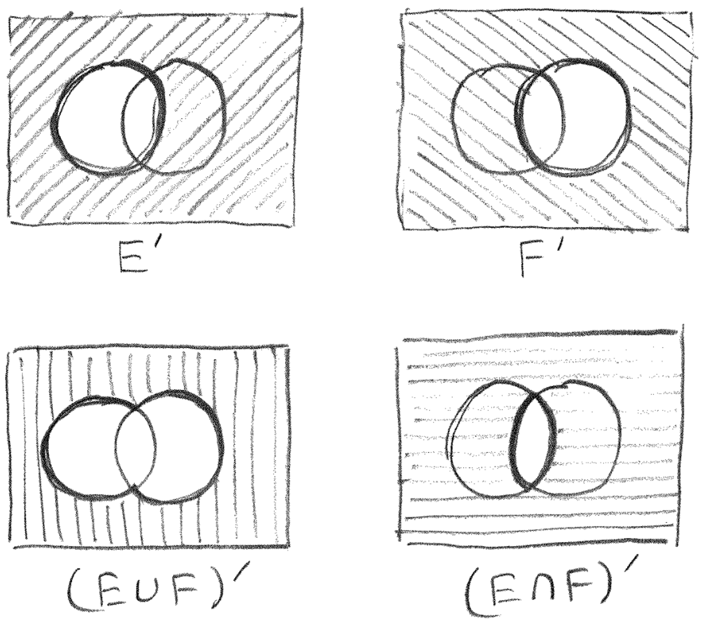
Finally, observe that $C \cap D = \{HTHT, THHT, TTHH\}$ so that

$$P(C \cap D) = \frac{\#(C \cap D)}{\#S} = \frac{3}{16}.$$

3. Draw Venn diagrams to verify *de Morgan's laws*: For all events $E, F \subseteq S$ we have

(a) $(E \cup F)' = E' \cap F'$,
(b) $(E \cap F)' = E' \cup F'$.

The proof follows from the following diagrams:



4. Suppose that a fair coin is flipped until heads appears. The sample space is

$$S = \{H, TH, TTH, TTTH, TTTTH, \ldots\}.$$

However these outcomes are **not equally likely**.

(a) Let E_k be the event {first H occurs on the kth flip}. Explain why $P(E_k) = 1/2^k$. [Hint: The event E_k consists of exactly one outcome. What is the probability of this outcome? You may assume that the coin flips are **independent**.]

(b) Recall the *geometric series* from Calculus:

$$1 + q + q^2 + \cdots = \frac{1}{1-q} \qquad \text{for all numbers } |q| < 1.$$

Use this fact to verify that the sum of all the probabilities equals 1:

$$\sum_{k=1}^{\infty} P(E_k) = 1.$$

(a) There is exactly one outcome in this event:

$$E_k = \{\underbrace{TTT\cdots T}_{k-1 \text{ times}} H\}.$$

Since the coin flips are fair and independent we have

$$\begin{aligned} P(E_k) &= P(\underbrace{TTT\cdots T}_{k-1 \text{ times}} H) \\ &= \underbrace{P(T)P(T)P(T)\cdots P(T)}_{k-1 \text{ times}} P(H) \\ &= \underbrace{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)}_{k-1 \text{ times}}\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k = \frac{1}{2^k}. \end{aligned}$$

(b) By substituting $q = 1/2$ into the geometric series we obtain

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots &= \frac{1}{1 - 1/2} \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots &= 2 \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots &= 2 - 1 \\ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots &= 1 \end{aligned}$$

and hence

$$\sum_{k=0}^{\infty} P(E_k) = \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

5. Suppose that $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.2$. Use this information to compute the following probabilities. A Venn diagram may be helpful.

(a) $P(A \cup B)$,
(b) $P(A \cap B')$,
(c) $P(A' \cup B')$.

(a) Using Inclusion-Exclusion for two events gives

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.2 = \boxed{0.7.}$$

(b) Using the Law of Total Probability gives

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ 0.3 &= 0.2 + P(A \cap B') \\ \boxed{0.1} &= P(A \cap B'). \end{aligned}$$

(c) Using de Morgan's Law and Complementary Events gives

$$P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - 0.2 = \boxed{0.8.}$$

6. Let X be a real number that is selected randomly from $[0, 1]$, i.e., the closed interval from zero to one. Use your intuition to assign values to the following probabilities:

(a) $P(X = 1/2)$,
(b) $P(0 \leq X \leq 1/3)$,
(c) $P(0 < X < 1/3)$,
(d) $P(1/2 < X \leq 3/4)$,
(e) $P(1/2 < X < 4/3)$.

(a) If all of the points in $[0, 1]$ are "equally likely," then since there are infinitely many points we must have

$$P(X = 1/3) = \frac{1}{\infty} = 0.$$

Maybe you're uncomfortable with this, but it's the least wrong answer I can think of. My intuition is that we roll a ball on a billiard table. After the ball comes to rest we measure the distance from the center of the ball to a fixed side of the table. In the real world there is no way that the ball will stop **exactly** one third of the way across the table. Any measurement can only be approximate.

(c) It seems reasonable that there is a 1/3 chance of landing in the left third of the interval:

$$P(0 < X < 1/3) = 1/3.$$

(b) If you agreed in part (a) that $P(X = 1/3) = P(X = 0) = 0$ then we must have

$$\begin{aligned} P(0 \leq X \leq 1/3) &= \cancel{P(X=0)} + P(0 < X < 1/3) + \cancel{P(X = 1/3)} \\ &= 0 + P(0 < X < 1/3) + 0 \\ &= P(0 < X < 1/3) = 1/3. \end{aligned}$$

(c) In general, the probability of landing in an interval should be the **length** of the interval. And we can just ignore the endpoints.

$$P(1/2 < X \leq 3/4) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

(d) It is impossible to get $1 < X < 4/3$, so we must have

$$\begin{aligned} P(1/2 < X < 4/3) &= P(1/2 < X \leq 1) + \cancel{P(1 < X < 4/3)} \\ &= 1/2 + 0 \\ &= 1/2. \end{aligned}$$

7. Consider a strange coin with $P(H) = p$ and $P(T) = q = 1 - p$. Suppose that you flip the coin n times and let X be the number of heads that you get. Find a formula for the probability $P(X \geq 1)$. [Hint: Observe that $P(X \geq 1) + P(X = 0) = 1$. Maybe it's easier to find a formula for $P(X = 0)$.]

There is only one way to get $X = 0$:

$$\text{“}X = 0\text{”} = \{\underbrace{TTT\cdots T}_{n \text{ times}}\}.$$

Then by independence we must have

$$\begin{aligned} P(X = 0) &= P(\underbrace{TTT\cdots T}_{n \text{ times}}) \\ &= \underbrace{P(T)P(T)P(T)\cdots P(T)}_{n \text{ times}} \\ &= \underbrace{qqq\cdots q}_{n \text{ times}} = q^n \end{aligned}$$

and hence $P(X \geq 1) = 1 - P(X = 0) = 1 - q^n$.

8. Suppose that you roll a pair of fair six-sided dice.

(a) Write down all elements of the sample space S. What is $\#S$? Are the outcomes equally likely? [Hopefully, yes.]

(b) Compute the probability of getting a “double six.” [Hint: Let $E \subseteq S$ be the subset of outcomes that correspond to getting a “double six.” Assuming that the outcomes of your sample space are equally likely, you can use the formula $P(E) = \#E/\#S$.]

(a) Let's suppose that one die is "blue" and the other is "red," so we can tell them apart. In other words, the outcome "12"="the blue die shows 1 and the red die shows 2" will differ from the outcome "21"="the blue die shows 2 and the red die shows 1." The the sample space is:

$$\begin{aligned} S = \{&11, 12, 13, 14, 15, 16 \\ &21, 22, 23, 24, 25, 26 \\ &31, 32, 33, 34, 35, 36 \\ &41, 42, 43, 44, 45, 46 \\ &61, 62, 63, 64, 65, 66\}. \end{aligned}$$

Independence and fairness suggest that for any outcome $ij \in S$ we must have $P(ij) = P(i)P(j) = (1/6)(1/6) = 1/36$. In other words, the 36 outcomes are equally likely.[1]

(b) Let $E =$"double six," so that $E = \{66\}$. Then we have

$$P(E) = \frac{\#E}{\#S} = \frac{1}{36}.$$

9. Analyze the Chevalier de Méré's two experiments:

(a) Roll a fair six-sided die 4 times and let X be the number of "sixes" that you get. Compute $P(X \geq 1)$. [Hint: You can think of a die roll as a "strange coin flip," where $H =$"six" and $T =$"not six." Use Problem 7.]

(b) Roll a pair of fair six-sided dice 24 times and let Y be the number of "double sixes" that you get. Compute $P(Y \geq 1)$. [Hint: You can think of rolling two dice as a "very strange coin flip," where $H =$"double six" and $T =$"not double six." Use Problems 7 and 8.]

(a) Roll a fair six-sided die and let $H =$"we get six," so that $P(H) = p = 1/6$ and $P(T) = q = 5/6$. Then according to Problem 7 we have

$$P(X \geq 1) = 1 - q^4 = 1 - \left(\frac{5}{6}\right)^4 = \boxed{51.77\%.}$$

(b) Roll a pair of fair six-sided dice and let $H =$"we get double six." Then from Problem 8 we know that $P(H) = p = 1/36$ and $P(T) = q = 35/36$ and from Problem 7 we find

$$P(Y \geq 1) = 1 - q^{24} = 1 - \left(\frac{35}{36}\right)^{24} = \boxed{49.14\%.}$$

[Remark: This agrees with the Chevalier's experimental evidence that $P(X \geq 1)$ is slightly greater than 50% and $P(Y \geq 1)$ is slightly less than 50%.]

[1] It's perfectly okay to consider the two dice as "unordered" or "uncolored." Then we will have $\#S = 21$. However, in this case the outcomes will **not** be equally likely, which makes the analysis much harder.