Math 224 **Spring 2022**
Homework 2 **Drew Armstrong**

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1. Tricks With the Binomial Theorem. For any real number x and integer $n \geq 0$, the Binomial Theorem tells us that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

Use this to prove the following identities:

(a) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

(b) $1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$

(c) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$

[Hint: Differentiate the polynomial and/or substitute various values of x.]

2. Counting License Plates. Find the total number of possible license plates with the following properties:

(a) Four letters followed by two digits.
(b) The same as (a), but letters and digits cannot be repeated.
(c) Four letters and two digits in any order. Repetition is okay. [Hint: First choose the positions for the letters and digits. Then count how many ways to fill them in.]

Remark: We use an alphabet with 26 letters.

3. Multinomial Probability. A fair eight-sided die has 2 sides painted red, 3 sides painted green and 3 sides painted blue. Suppose you roll the die 5 times and let R, G, B be the number of times you get red, blue and green, respectively. Compute the following probabilities:

(a) $P(R = 2, G = 1, B = 2)$
(b) $P(R \geq 1)$ [Hint: Think of the die as a coin with "heads"="you get red".]
(c) $P(R = G)$ [Hint: If $R = G$ then what are the possible values of R, G, B?]

4. Hypergeometric Probability. A deck of cards contains 26 red cards and 26 black cards. Choose 5 cards at random and let R be the number of red cards that you get.

(a) What is the size of the sample space?
(b) How many outcomes are there with $R = k$? [Hint: Your answer will involve k.]
(c) Use parts (a) and (b) to compute the probabilities $P(R = k)$ for $k = 0, 1, 2, 3, 4, 5$. [Hint: Use a calculator.]

5. The Birthday Problem. We choose n people at random and record their birthdays as a sequence of n numbers from the set $\{1, 2, \ldots, 365\}$. (We ignore leap years.) Assume that each birthday is equally likely.

(a) What is the size of the sample space?
(b) Compute the probability that no two people have the same birthday. [Hint: Count the sequences with no repeated numbers. Your answer will involve n.]

(c) Let $f(n)$ be the probability that in a random group of n people there exist two people with the same birthday. Note that $f(1) = 0\%$ and $f(366) = 100\%$. Find the smallest n such that $f(n) > 50\%$. [Hint: Use part (b) to find a formula for $f(n)$. Then use a calculator to plug in various values of n. The answer is surprisingly small.]

6. Bayes' Theorem. A random person is tested for a certain disease. Consider the events

$$D = \text{the person has the disease,}$$
$$T = \text{the test returns positive.}$$

Assume that the disease prevalence is $P(D) = 1\%$. Assume that the test has false positive rate $P(T|D') = 0.1\%$ and false negative rate $P(T'|D) = 5\%$.

(a) Find the true negative rate $P(T'|D')$ and the true positive rate $P(T|D)$. [Hint: Easy.]

(b) Find the probability $P(T)$ that the test returns positive. [Hint: Use the definition of conditional probability, $P(A \cap B) = P(A)P(B|A)$, and the Law of Total Probability, which tells us that $P(T) = P(D \cap T) + P(D' \cap T)$.]

(c) Compute the backwards probabilities $P(D|T)$ and $P(D'|T')$. [Hint: The definition of conditional probability says that $P(D|T) = P(D \cap T)/P(T)$.]