Math 224 **Spring 2022**
Homework 1 **Drew Armstrong**

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1. A Fair Coin.

(a) Draw Pascal's triangle down to the sixth row. (The first row consists of a single 1.)
(b) Suppose a **fair** coin is flipped 5 times and let X be the number of heads you get. Use part (a) to compute the probabilities $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5$.
(c) Use your answer from part (b) to compute the following probabilities:

$$P(X \geq 4), \quad P(X \text{ is odd}), \quad P(X \geq 1).$$

(a): Here is Pascal's Triangle:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Remark: To be consistent with the course notes, I should have called this the "fifth row".

(b): Since $2^5 = 32$ we have the following table of probabilities:

k	0	1	2	3	4	5
$P(X = k)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

(c): For each event we add up the corresponding probabilities:

$$\begin{aligned} P(X \geq 4) &= P(X = 4) + P(X = 5) = 5/32 + 1/32 = 6/32, \\ P(X \text{ is odd}) &= P(X = 1) + P(X = 3) + P(X = 5) = 5/32 + 10/32 + 1/32 = 16/32, \\ P(X \geq 1) &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ &= 5/32 + 10/32 + 10/32 + 5/32 + 1/32 = 31/32. \end{aligned}$$

Alternatively, it is easier to compute $P(X \geq 1)$ as follows:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 1/32 = 31/32.$$

2. A Biased Coin. Consider a general coin with $P(H) = p$ and $P(T) = q$, where p and q are any real numbers satisfying $p, q \geq 0$ and $p + q = 1$.

(a) Suppose you flip the coin 5 times and let X be the number of heads you get. Use Problem 1(a) to compute the probabilities $P(X = k)$ for $k = 0, 1, 2, 3, 4, 5$. [Hint: Your answers will involve the unknown constants p and q.]
(b) Compute the probability $P(X \geq 1)$. [Hint: It is easier to compute $P(X = 0)$ and then use the fact that $P(X = 0) + P(X \geq 1) = 1$. Your answer will contain p and/or q.]

(a): The general formula says that

$$P(X = k) = \binom{5}{k} p^k q^{5-k},$$

where $\binom{5}{k}$ is the kth entry in the fifth row of Pascal's triangle. From Problem 1(a) we obtain the following table of probabilities:

k	0	1	2	3	4	5
$P(X = k)$	q^5	$5pq^4$	$10p^2q^3$	$10p^3q^2$	$5p^4q$	p^5

Remark: These probabilities add to 1 because of the algebraic identity

$$\begin{aligned} 1 &= q + p \\ 1 &= (q+p)^5 \\ 1 &= q^5 + 5pq^4 + 10p^2q^3 + 10p^3q^2 + 5p^4q + p^5. \end{aligned}$$

We will see the details when we discuss the Binomial Theorem.

(c): From part (b) we obtain

$$P(X \geq 1) = 1 - P(X = 0) = 1 - q^5.$$

3. Working With Events. Suppose that you flip a **fair** coin 4 times. We we record each possible outcome as a sequence of H's and T's, such as "$THHT$".

(a) List the elements of the sample space S. What is $\#S$? Since the coin is fair, we can assume that each of these outcomes is equally likely, so the probably of any event $E \subseteq S$ is given by $P(E) = \#E/\#S$.

(b) List the elements in each of the following events:

$$\begin{aligned} A &= \{\text{exactly 2 heads}\}, \\ B &= \{\text{an even number of heads}\}, \\ C &= \{\text{heads on the first flip, anything on the other flips}\}. \end{aligned}$$

(c) Use parts (a) and (b) to compute the following probabilities:

$$P(A), \quad P(B), \quad P(C), \quad P(A \cap C), \quad P(A \cup B).$$

[Hint: The *intersection* $A \cap C$ is the set of outcomes that are in A **and** in C. The *union* $A \cup B$ is the set of outcomes that are in A **or** in B, or in both.]

(a): The sample space is

$$\begin{aligned} S = \{&TTTT, \\ &HTTT, THTT, TTHT, TTTH, \\ &HHTT, HTHT, HTTH, THHT, THTH, TTHH, \\ &HHHT, HHTH, HTHH, THHH, \\ &HHHH\}. \end{aligned}$$

Hence we have $\#S = 16$. This is also $\#S = 2^4$.

(b): The events are

$$
\begin{aligned}
A &= \{\text{exactly 2 heads}\} \\
&= \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\},
\end{aligned}
$$

$$
\begin{aligned}
B &= \{\text{an even number of heads}\} \\
&= \{TTTT, \\
&\quad\; HHTT, HTHT, HTTH, THHT, THTH, TTHH, \\
&\quad\; HHHH\},
\end{aligned}
$$

$$
\begin{aligned}
C &= \{\text{heads on the first flip, anything on the other flips}\} \\
&= \{HTTT, \\
&\quad\; HHTT, HTHT, HTTH, \\
&\quad\; HHHT, HHTH, HTHH, \\
&\quad\; HHHH\}.
\end{aligned}
$$

Remark: I include the outcome $TTTT$ in B because zero is an even number.

(c): Since the outcomes are equally likely we can use the formula $P(E) = \#E/\#S$ to obtain

$$
\begin{aligned}
P(A) &= \#A/\#S = 6/16, \\
P(B) &= \#B/\#S = 8/16, \\
P(C) &= \#C/\#S = 8/16.
\end{aligned}
$$

To compute $P(A \cap C)$ and $P(A \cap B)$ we will count the corresponding outcomes. First we have

$$A \cap C = \{HHTT, HTHT, HTTH\},$$

and hence $P(A \cap C) = \#(A \cap C)/\#S = 3/16$. Next we observe that $A \cup B = B$ because every element of A is already in B, hence we have

$$P(A \cup B) = P(B) = 8/16.$$

Remark: Several students tried to use the formulas $P(A \cap C) = P(A)P(C)$ and $P(A \cap B) = P(A)P(B)$. **These formulas are not always true!** If events $E, F \subseteq S$ satisfy $P(E \cap F) = P(E)P(F)$ then we say that E and F are independent, but not all pairs of events are independent. If happens by accident that A and C are independent, but A and B are not. We can see this by observing that $A \cap B = A$[1] so that $P(A \cap B) = P(A) = 6/16$. Hence

$$P(A \cap B) = 6/16 \neq (6/16)(8/16) = P(A)P(B).$$

In general there is no easy way to predict when two events are independent.

4. Rolling a Pair of Fair Dice. Suppose that you roll a pair of **fair** six-sided dice, each with sides labeled $\{1, 2, 3, 4, 5, 6\}$. We will record the outcome that "the first die shows i and the second die shows j" with the symbol "ij".

[1]For any events satisfying $A \subseteq B$ we must have $A \cap B = A$ and $A \cup B = B$.

(a) List the elements of the sample space S. What is $\#S$? Since the dice are fair, we can assume that each of these outcomes is equally likely, so the probability of any event $E \subseteq S$ is $\#E/\#S$.
(b) Compute the probability of getting a "double six", i.e., a 6 on both dice.
(c) Let X be the sum of the two numbers that show up on the dice. Use part (a) to compute the probabilities $P(X = k)$ for $X = 2, 3, 4, \ldots, 12$. [Hint: Count the number of outcomes corresponding to each value of X.]

(a): I will list the outcomes in a 6×6 array:

$$\begin{aligned} S = \{ &11, 12, 13, 14, 15, 16 \\ &21, 22, 23, 24, 25, 26 \\ &31, 32, 33, 34, 35, 36 \\ &41, 42, 43, 44, 45, 46 \\ &51, 52, 53, 54, 55, 56 \\ &61, 62, 63, 64, 65, 66\}. \end{aligned}$$

Hence we have $\#S = 6^2 = 36$. Each of these outcomes is equally likely, so for any event $E \subseteq S$ we will have $P(E) = \#E/\#S = \#E/36$.

Remark: Maybe you don't want to distinguish between the two dice. For example, maybe you want to consider the outcomes 56 and 65 as the same. This is technically fine but it makes the problem much more difficult because then the outcomes are not equally likely. For example, we know from experiment that the outcome "get 5 and 6 in any order" is twice as likely as the outcome "double six".

(b): There is only one way to get "double six", so we have $P(\text{double six}) = 1/36$.

(c): Here are the events corresponding to different values of k:

$$\begin{aligned} \text{“}X = 2\text{”} &= \{11\}, \\ \text{“}X = 3\text{”} &= \{12, 21\}, \\ \text{“}X = 4\text{”} &= \{13, 22, 31\}, \\ \text{“}X = 5\text{”} &= \{14, 23, 32, 41\}, \\ \text{“}X = 6\text{”} &= \{15, 24, 33, 42, 51\}, \\ \text{“}X = 7\text{”} &= \{16, 25, 34, 43, 52, 61\}, \\ \text{“}X = 8\text{”} &= \{26, 35, 44, 53, 62\}, \\ \text{“}X = 9\text{”} &= \{36, 45, 54, 63\}, \\ \text{“}X = 10\text{”} &= \{46, 55, 64\}, \\ \text{“}X = 11\text{”} &= \{56, 65\}, \\ \text{“}X = 12\text{”} &= \{66\}. \end{aligned}$$

Thus we obtain the following table of probabilities:

k	2	3	4	5	6	7	8	9	10	11	12
$P(X = k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

5. The Chevalier's Problems.

(a) Consider a general coin with $P(H) = p$ and $P(T) = q$. Suppose you flip the coin n times and let X be the number of heads you get. Compute $P(X \geq 1)$. [Hint: Compare with Problem 2(b). Your answer may contain the letters n, p, q.]

(b) Suppose that you roll a fair six-sided die 4 times and let X be the number of "sixes" that you get. Compute $P(X \geq 1)$. [Hint: Think of a die roll as a "strange coin flip" with $H =$"you get a six" and $T =$"you don't get six". What are p and q in this case?]

(c) Suppose that you roll a pair fair six-sided dice 24 times and let X be the number of "double sixes" that you get. Compute $P(X \geq 1)$. [Hint: Think of one roll of the dice as a "very strange coin flip" with $H =$"you get a double six" and $T =$"you don't get double six". What are p and q in this case?]

(a): The general formula is $P(X = k) = \binom{n}{k} p^k q^{n-k}$. Since the first entry in the nth row of Pascal's triangle is $1 = \binom{n}{0}$ we obtain

$$P(X = 0) = 1p^0q^n = q^n$$

and hence

$$P(X \geq 1) = 1 - P(X = 0) = 1 - q^n.$$

This is the probability of getting at least one head in n coin flips. If $q < 1$ (i.e., if heads is not impossible) then we have $q^n \to 0$ as $n \to 1$, hence $P(X \geq 1) \to 1$. In other words: If we keep flipping the coin forever then we are guaranteed to get heads at some point.

(b): Think of a die roll as a coin flip with $H =$"get six" and $T =$"don't get six". In this case we have $p = P(H) = 1/6$ and $q = P(T) = 5/6$. If X is the number of sixes obtained in 4 rolls of the die, then from part (a) we have

$$P(X \geq 1) = 1 - q^4 = 1 - (5/6)^4 \approx 51.77\%.$$

(c) Think of a double die roll as a coin flip with $H =$"get double six" and $T =$"don't get double six". Then from Problem 4(b) we have $p = P(H) = 1/36$ and $q = P(T) = 35/36$. If X is the number of double sixes obtained in 24 rolls of the dice, then from part (a) we have

$$P(X \geq 1) = 1 - q^{24} = 1 - (35/36)^{24} \approx 49.14\%.$$

Bonus Content: It is interesting to consider the case of 3 dice. This experiment has $6^3 = 216$ possible outcomes, each of which is equally likely. Think of a roll of the dice as a coin flip with $H =$"get triple six" and $T =$"don't get triple six", so that $p = P(H) = 1/216$ and $q = P(T) = 215/216$. If you roll the dice n times and let X be the number of triple sixes you get, then from part (a) we have

$$P(\text{at least one triple six in } n \text{ rolls of three dice}) = P(X \geq 1) = 1 - q^n = 1 - (215/216)^n.$$

This probability ranges from 0% when $n = 0$ to 100% as $n \to \infty$. (You are guaranteed to get a triple six if you keep rolling the dice forever.) Hence for some value of n it must cross 50%. When does this happen? Let's set $P(X \geq 1) = 1/2$ and solve for n:

$$\begin{aligned}
1 - (215/216)^n &= 1/2 \\
(215/216)^n &= 1/2 \\
n \log(215/216) &= \log(1/2) \\
n &= \log(1/2)/\log(215/216) \approx 149.37.
\end{aligned}$$

But n must be a whole number. The two closes values of n give

$$P(X \geq 1) \approx 49.91\% \text{ when } n = 149,$$
$$P(X \geq 1) \approx 50.15\% \text{ when } n = 150.$$

In other words: If you roll three dice 149 times then you have a 49.91% chance of getting "triple six" at least once. If you roll the dice 150 times then you have a 50.15% chance of getting "triple six" at least once.