Math 224 **Spring 2018**
Homework 5 **Drew Armstrong**

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1. Let U be the uniform random variable on the interval $[2,5]$. Compute the following:

$$P(U=0),\quad P(U=3),\quad P(0<U<3),\quad P(3<U<4.5),\quad P(3\leq U\leq 4.5).$$

2. Let X be a continuous random variable with pdf defined as follows:

$$f_X(x)=\begin{cases} c\cdot x^2 & \text{if } 0\leq x\leq 1,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Compute the value of the constant c.
(b) Find the mean $\mu=E[X]$ and standard deviation $\sigma=\sqrt{\text{Var}(X)}$.
(c) Compute the probability $P(\mu-\sigma\leq X\leq\mu+\sigma)$.
(d) Draw the graph of f_X, showing the interval $\mu\pm\sigma$ in your picture.

3. Let Z be a standard normal random variable, which is defined by the following pdf:

$$n(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}.$$

Let $\Phi(z)$ be the associated cdf (cumulative density function), which is defined by

$$\Phi(z)=P(Z\leq z)=\int_{-\infty}^{z} n(x)\,dx.$$

Use the attached table to compute the following probabilities:

(a) $P(0<Z<0.5)$,
(b) $P(Z<-0.5)$,
(c) $P(Z>1),\ P(Z>2),\ P(Z>3)$.
(d) $P(|Z|<1),\ P(|Z|<2),\ P(|Z|<3)$,

4. Continuing from Problem 3, use the attached table to find numbers $c,d\in\mathbb{R}$ solving the following equations:

(a) $P(Z>c)=P(|Z|>d)=2.5\%$,
(b) $P(Z>c)=P(|Z|>d)=5\%$,
(c) $P(Z>c)=P(|Z|>d)=10\%$.

5. Let $X\sim N(\mu,\sigma^2)$ be a normal random variable with mean μ and variance σ^2. Let $\alpha,\beta\in\mathbb{R}$ be any constants such that $\alpha\neq 0$ and consider the random variable

$$Y=\alpha X+\beta.$$

(a) Show that $E[Y]=\alpha\mu+\beta$ and $\text{Var}(Y)=\alpha^2\sigma^2$.
(b) Show that Y has a normal distribution $N(\alpha\mu+\beta,\alpha^2\sigma^2)$. In other words, show that for all real numbers $y_1\leq y_2$ we have

$$P(y_1\leq Y\leq y_2)=\int_{y_1}^{y_2}\frac{1}{\sqrt{2\pi\alpha^2\sigma^2}}\cdot e^{-[y-(\alpha\mu+\beta)]^2/2\alpha^2\sigma^2}\,dy.$$

[Hint: For all $x_1\leq x_2$ you may assume that

$$P(x_1\leq X\leq x_2)=\int_{x_1}^{x_2}\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-(x-\mu)^2/2\sigma^2}\,dx.$$

Now use the substitution $y = \alpha x + \beta$.]

It follows from this problem that $Z = (X - \mu)/\sigma = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ has a **standard** normal distribution. That is extremely useful.

6. The average weight of a bag of chips from a certain factory is 150 grams. Assume that the weight is normally distributed with a standard deviation of 12 grams.

(a) What is the probability that a given bag of chips has weight greater than 160 grams?

(b) Collect a random sample of 10 bags of chips and let Y be the number that have weight greater than 160 grams. Compute the probability $P(Y \leq 2)$.

7. Let $X_1, X_2, \ldots, X_{15}$ be independent and identically distributed (i.i.d.) random variables. Suppose that each X_i has pdf defined by the following function:

$$f(x) = \begin{cases} \frac{3}{2} \cdot x^2 & \text{if } -1 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute $E[X_i]$ and $\mathrm{Var}(X_i)$.

(b) Consider the sum $Y = X_1 + X_2 + \cdots + X_{15}$. Use part (a) to compute $E[Y]$ and $\mathrm{Var}(Y)$.

(c) The Central Limit Theorem says that Y is approximately normal. Use this fact to estimate the probability $P(-0.3 \leq Y \leq 0.5)$.

8. Suppose that $n = 48$ seeds are planted and suppose that each seed has a probability $p = 75\%$ of germinating. Let X be the number of seeds that germinate and use the Central Limit Theorem to estimate the probability $P(35 \leq X \leq 40)$ that between 35 and 40 seeds germinate. Don't forget to use a continuity correction.

9. Suppose that a six-sided die is rolled 24 times and let X_k be the number that shows up on the kth roll. Let $\overline{X} = (X_1 + X_2 + \cdots + X_{24})/24$ be the average of the numbers that show up.

(a) Assuming that the die is fair, compute the expected value and variance:

$$E\left[\overline{X}\right] \qquad \text{and} \qquad \mathrm{Var}\left(\overline{X}\right).$$

(b) Assuming that the die is fair, use the Central Limit Theorem to estimate the probability $P\left(\overline{X} \geq 4\right)$.

(c) Suppose you roll an unknown six-sided die 24 times and get an average value of 4.

Is the die fair?

In other words: Let H_0 be the hypothesis that the die is fair. Should you reject this hypothesis at the 5% level of significance?

Standard Normal Probabilities

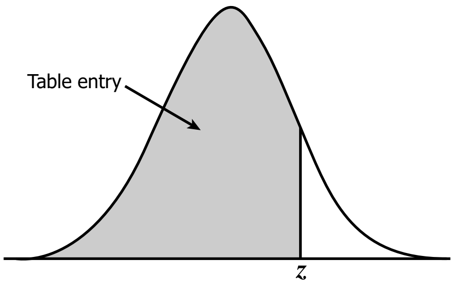

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998