Math 224 **Spring 2018**
Homework 3 **Drew Armstrong**

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1. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin until the first head shows up and let X be the number of flips you made. The probability mass function and support of this *geometric random vabiable* are given by

$$P(X = k) = q^{k-1}p \qquad \text{and} \qquad S_X = \{1, 2, 3, \ldots\}.$$

(a) Use the geometric series $1 + q + q^2 + \cdots = (1-q)^{-1}$ to show that

$$\sum_{k \in S_X} P(X = k) = 1.$$

(b) Differentiate the geometric series to get $0 + 1 + 2q + 3q^2 + \cdots = (1-q)^{-2}$ and use this series to show that

$$E[X] = \sum_{k \in S_X} k \cdot P(X = k) = \frac{1}{p}.$$

(c) Application: Start rolling a fair 6-sided die. On average, how long do you have to wait until you see "1" for the first time?

2. There are 2 red balls and 4 green balls in an urn. Suppose you grab 3 balls without replacement and let X be the number of red balls you get.

(a) What is the support of this random variable?
(b) Draw a picture of the probability mass function $f_X(k) = P(X = k)$.
(c) Compute the expected value $E[X]$. Does the answer make sense?

3. Roll a pair of fair 6-sided dice and consider the following random variables:

$$X = \text{the number that shows up on the first roll,}$$
$$Y = \text{the number that shows up on the second roll.}$$

(a) Write down all elements of the sample space S.
(b) Compute the probability mass function for the sum $f_{X+Y}(k) = P(X + Y = k)$ and draw the probability histogram.
(c) Compute the expected value $E[X + Y]$ in two different ways.
(d) Compute the probability mass function for the difference $f_{X-Y}(k) = P(X - Y = k)$ and draw the probability histogram.
(e) Compute the expected value $E[X - Y]$ in two different ways.
(f) Compute the probability mass function for the absolute value of the difference

$$f_{|X-Y|}(k) = P(|X - Y| = k)$$

and draw the probability histogram.
(e) Compute the expected value $E\left[|X - Y|\right]$. This time there is only one way to do it.

4. Let X be a random variable satisfying

$$E[X + 1] = 3 \qquad \text{and} \qquad E[(X + 1)^2] = 10.$$

Use this information to compute the following:

$$\mathrm{Var}(X + 1), \quad E[X], \quad E[X^2] \quad \text{and} \quad \mathrm{Var}(X).$$

5. Let X be a random variable with mean $E[X] = \mu$ and variance $\text{Var}(X) = \sigma^2 \neq 0$. Compute the mean and variance of the random variable Y defined by

$$Y = \frac{X - \mu}{\sigma}.$$

6. Let X be the number of strangers you must talk to until you find someone who shares your birthday. (Assume that each day of the year is equally likely and ignore February 29.)

(a) Find the probability mass function $P(X = k)$.
(b) Find the expected value $\mu = E[X]$.
(c) Find the *cumulative mass function* $P(X \leq k)$. Hint: If X is a geometric random variable with pmf $P(X = k) = q^{k-1}p$, use the geometric series to show that

$$P(X \leq k) = 1 - P(X > k) = 1 - \sum_{i=k+1}^{\infty} q^{i-1}p = 1 - q^k.$$

(d) Use part (c) to find the probability $P(\mu - 50 \leq X \leq \mu + 50)$ that X falls within ± 50 of the expected value. Hint:

$$P(\mu - 50 \leq X \leq \mu + 50) = P(X \leq \mu + 50) - P(X \leq \mu - 50 - 1).$$

7. I am running a lottery. I will sell 10 tickets, each for a price of \$1. The person who buys the winning ticket will receive a cash prize of \$5.

(a) If you buy one ticket, what is the expected value of your profit?
(b) If you buy two tickets, what is the expected value of your profit?
(c) If you buy n tickets $(0 \leq n \leq 10)$, what is the expected value of your profit? Which value of n maximizes your expected profit?

[Remark: Profit equals prize money minus cost of the tickets.]

8. Consider a coin with $P(H) = p$ and $P(T) = q$. Flip the coin n times and let X be the number of heads you get. In this problem you will give a bad proof that $E[X] = np$.

(a) Use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.
(b) Complete the following computation:

$$\begin{aligned}
E[X] &= \sum_{k=0}^{n} k \cdot P(X = k) \\
&= \sum_{k=1}^{n} k \cdot P(X = k) \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k q^{n-k} \\
&= \cdots
\end{aligned}$$