Math 224 **Fall 2021**
Homework 5 **Drew Armstrong**

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1. Let U be the uniform random variable on the interval $[2,5]$. Compute the following:

$$P(U<3),\quad P(3<U<6),\quad \mu=E[U],\quad \sigma^2=\text{Var}(U),\quad P(\mu-\sigma<U<\mu+\sigma).$$

2. Let X be a continuous random variable with the following density:

$$f_X(x)=\begin{cases} c\cdot\sin(x) & 0\leq x\leq\pi,\\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the value of the constant c.
(b) Compute $\mu=E[X]$ and $\sigma^2=\text{Var}(X)$.[1]
(c) Compute $P(\mu-\sigma<X<\mu+\sigma)$.
(d) Draw a picture of the whole situation.

3. Mean and Variance of a Normal Density. Let $X\sim N(\mu,\sigma^2)$ and $Z\sim N(0,1)$. In other words, suppose that X and Z have the following densities:

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-(x-\mu)^2/2\sigma^2}\quad\text{and}\quad f_Z(z)=\frac{1}{\sqrt{2\pi}}\cdot e^{-z^2/2}.$$

(a) Compute the expected value $E[Z]$. [Hint: Substitute $u=-z^2/2$.]
(b) Compute the second moment $E[Z^2]$ and the variance $\text{Var}(Z)$. [Hint: Use integration by parts with $u=-z$ and $v=e^{-z^2/2}$. You may assume that $\int f_Z(z)\,dz=1$.]
(c) Use parts (a) and (b) to compute $E[X]$ and $\text{Var}(X)$. [Hint: We showed in class that $(X-\mu)/\sigma$ and Z have the same density.]

4. Let $Z\sim N(0,1)$ so that $P(Z\leq z)=\Phi(z)$. Use the attached table to compute the following probabilities:

(a) $P(Z<-0.5)$
(b) $P(0.33<Z<1.25)$
(c) $P(Z>1)$, $P(Z>2)$, $P(Z>3)$
(d) $P(|Z|<1)$, $P(|Z|<2)$, $P(|Z|<3)$

5. De Moivre-Laplace Consider a coin with $p=P(H)=1/3$. Suppose that you flip the coin 100 times and let X be the number of times you get heads. Use the de Moive-Laplace theorem to compute the probability

$$P(30\leq X\leq 35).$$

Don't forget to use a continuity correction.

6. Central Limit Theorem. Let $X_1,X_2,X_3,\ldots,X_{1000}$ be a sequence of iid random variables, each with mean $\mu=600$ and variance $\sigma^2=40$. Consider the sample mean

$$\overline{X}=(X_1+X_2+\cdots+X_{1000})/1000.$$

Use the central limit theorem to estimate the probability that $\overline{X}$ falls between 599.9 and 600.1. [Remark: You should not use a continuity correction because the variables X_i are not necessarily discrete.]

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[1] Hint: $\int x\sin(x)\,dx=\sin(x)-x\cos(x)$ and $\int x^2\sin(x)\,dx=2\cos(x)+2x\sin(x)-x^2\cos(x)$.

7. Sample Variance. Let $X_1, X_2, \ldots, X_n$ be a sequence of iid random variables, each with the same mean $\mu = E[X_i]$ and variance $\sigma^2 = \text{Var}(X_i)$. We define the *sample mean*

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and the *sample variance*

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

(a) Show that $E[X_i^2] = \mu^2 + \sigma^2$.

(b) Show that $E[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \sigma^2/n$ and use these to show that

$$E[\overline{X}^2] = \mu^2 + \sigma^2/n.$$

(c) Show that

$$\sum_{i=1}^{n}(X_i - \overline{X})^2 = \left(\sum_{i=1}^{n} X_i^2\right) - n\overline{X}^2.$$

[Hint: $\overline{X}\sum_i X_i = n\overline{X}^2$.]

(d) Use parts (a),(b),(c) and the linearity of expectation to show that $E[S^2] = \sigma^2$. This explains why we use $n-1$ instead of n in the denominator of the sample variance.