Math 224 **Fall 2021**
Homework 4 **Drew Armstrong**

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1. The St Petersburg Paradox. I am running a game. I will let you flip a fair coin until you get heads. If the first head shows up on the kth flip then I will give you r^k dollars.

(a) Compute your expected winnings when $r = 1$.
(b) Compute your expected winnings when $r = 1.5$.
(c) Compute your expected winnings when $r = 2$. Does this make any sense? How much would you be willing to pay me to play this game?

[Hint: Use the geometric series.]

2. Let X be a random variable satisfying $E[X] = 1$ and $E[X^2] = 2$. Use this to compute

(a) $\text{Var}(X)$
(b) $E[(X+1)^2]$
(c) $\text{Var}(2X+3)$

3. Standardization. Let X be a random variable with $E[X] = \mu_X$ and $\text{Var}(X) = \sigma_X^2$ and consider the random variable

$$Z = \frac{X - \mu_X}{\sigma_X}.$$

(a) Use the linearity of expectation to compute $E[Z]$.
(b) Use the general properties of variance to compute $\text{Var}(Z)$.

4. Consider a fair six-sided die with sides labeled $\{1,2,3,4,5,6\}$. Roll the die twice and let

$$\begin{aligned} X &= \text{the number you get on the first roll,} \\ Y &= \text{the number you get on the second roll,} \\ Z &= X + Y. \end{aligned}$$

Compute the variances $\text{Var}(X)$, $\text{Var}(Y)$, $\text{Var}(Z)$ and the covariances $\text{Cov}(X,Y)$, $\text{Cov}(X,Z)$.

5. Let $X, Y : S \to \mathbb{R}$ be random variables with the following joint distribution table:

$X \setminus Y$	1	2	3	
1	1/21	5/21	3/21	9/21
2	4/21	2/21	6/21	12/21
	5/21	7/21	9/21	

How to read the table: We have $S_X = \{1,2\}$ and $S_Y = \{1,2,3\}$. The entries in the right column are $P(X = k)$, the entries in the bottom row are $P(Y = \ell)$ and the entries inside the table are $P(X = k, Y = \ell)$.

(a) Use the table to compute $P(X + Y \geq 4)$.
(b) Use the table to compute $E[X]$ and $E[Y]$.
(c) Use the table to compute $E[XY]$ and $\text{Cov}(X,Y)$.

6. Uncorrelated Does Not Imply Independent. We say that random variables $X, Y : S \to \mathbb{R}$ are independent if $P(X = k, Y = \ell) = P(X = k)P(Y = \ell)$ for all possible values $k, \ell \in \mathbb{R}$. This property implies that $E[XY] = E[X]E[Y]$ and hence $\text{Cov}(X, Y) = 0$. On the other hand, the identity $\text{Cov}(X, Y) = 0$ does not necessarily imply that X and Y are independent. Consider the following example:

$X \setminus Y$	-1	0	1	
-1	0	0	1/4	1/4
0	1/2	0	0	1/2
1	0	0	1/4	1/4
	1/2	0	1/2	

(a) Explain why these X and Y are **not** independent.
(b) Use the table to show that $\text{Cov}(X, Y) = 0$.

7. Multinomial Covariance. Suppose that a fair s-sided die is rolled n times, and let X_i be the number of times that the ith face shows up.

(a) Compute $\text{Var}(X_i)$ for any i. [Hint: Think of each roll as a coin flip with $H =$ "you get side i" and $T =$ "you don't get side i". Use the formula for variance of a binomial.]
(b) Compute $\text{Var}(X_i + X_j)$ for any $i \neq j$. [Hint: Think of each roll as a coin flip with $H =$ "you get side i or j" and $T =$ "you get some other side".]
(c) Combine (a), (b) to compute $\text{Cov}(X_i, X_j)$. Simplify your formula as much as possible.