Math 224 — **Fall 2021**
Homework 3 — **Drew Armstrong**

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1. Function of a Random Variable. Let $X : S \to \mathbb{R}$ be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be an ordinary function. Then the composition $g(X)$ [do X first, then do g] is another random variable and we have the following formula:

$$E[g(X)] = \sum_{k \in S_X} g(k) \cdot P(X = k).$$

Now suppose that X is the number of heads obtained in two flips of a fair coin. Use this formula to compute the following expected values:

(a) $E[X + 1]$
(b) $E[X^2]$
(c) $E[2^X]$

First we write down the probability mass function:

k	0	1	2
$P(X = k)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Now we apply the formula for each different function $g(x)$.

(a): If $g(x) = x + 1$ then we get

$$\begin{aligned} E[X+1] &= E[g(X)] \\ &= \sum_k g(k) \cdot P(X = k) \\ &= \sum_k (k+1) \cdot P(X = k) \\ &= (0+1)\frac{1}{4} + (1+1)\frac{1}{2} + (2+1)\frac{1}{4} = 2. \end{aligned}$$

Alternatively, since X is binomial with $n = 2$ and $p = 1/2$ we know that $E[X] = np = 1$. Then we can use the linearity of expectation:

$$E[X + 1] = E[X] + E[1] = 1 + 1 = 2.$$

(b): If $g(x) = x^2$ then we get

$$\begin{aligned} E[X^2] &= E[g(X)] \\ &= \sum_k g(k) \cdot P(X = k) \\ &= \sum_k k^2 \cdot P(X = k) \\ &= 0^2\frac{1}{4} + 1^2\frac{1}{2} + 2^2\frac{1}{4} = \frac{3}{2}. \end{aligned}$$

(b): If $g(x) = 2^x$ then we get

$$E[X^2] = E[g(X)]$$

$$
\begin{aligned}
&= \sum_k g(k) \cdot P(X = k) \\
&= \sum_k 2^k \cdot P(X = k) \\
&= 2^0 \frac{1}{4} + 2^1 \frac{1}{2} + 2^2 \frac{1}{4} = \frac{9}{4}.
\end{aligned}
$$

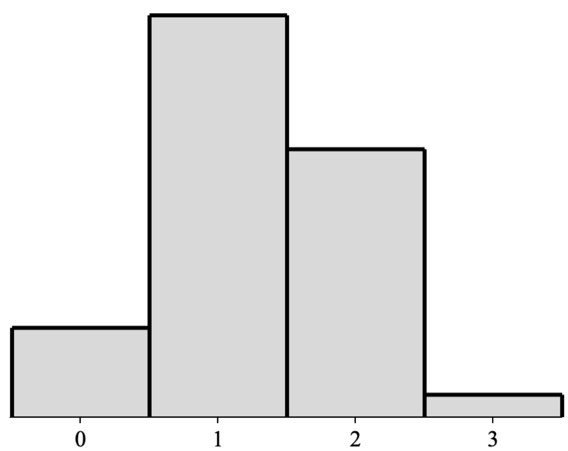
2. An urn contains 3 red balls and 4 green balls. Suppose you grab 3 balls without replacement and let R be the number of red balls that you get.

(a) Find a formula for the pmf $P(R = k)$ and draw the probability histogram.
(b) Compute the expected value $E[R]$.

(a): This is a hypergeometric random variable with $P(R = k) = \binom{3}{k}\binom{4}{3-k}/\binom{7}{3}$. Here is a table:

k	0	1	2	3
$P(R = k)$	$\frac{\binom{3}{0}\binom{4}{3}}{\binom{7}{3}} = \frac{4}{35}$	$\frac{\binom{3}{1}\binom{4}{2}}{\binom{7}{3}} = \frac{18}{35}$	$\frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = \frac{12}{35}$	$\frac{\binom{3}{3}\binom{4}{0}}{\binom{7}{3}} = \frac{1}{35}$

And here is a histogram:



(b): The expected value is

$$
\begin{aligned}
E[R] &= 0 \cdot P(R = 0) + 1 \cdot P(R = 1) + 2 \cdot P(R = 2) + 3 \cdot P(R = 3) \\
&= 0\frac{4}{35} + 1\frac{18}{35} + 2\frac{12}{35} + 3\frac{1}{35} = 9/7 \approx 1.3
\end{aligned}
$$

Remark: This answer makes sense. If $3/7$ of the balls in the urn are red and we grab n balls then we expect to get $(3/7)n$ red balls. When $n = 3$ this gives $(3/7)3 = 9/7$. In general, if an urn contains r red and g balls and if we grab n, then we expect to get $nr/(r + g)$ red balls. We proved this formula in class by expressing R as a sum of n Bernoulli random variables and using the linearity of expectation.

3. A fair four-sided die has sides labeled $\{1, 2, 3, 4\}$. Suppose you roll the die twice and consider the following random variables:

$$
\begin{aligned}
X &= \text{the number that shows up on the first roll,} \\
Y &= \text{the number that shows up on the second roll.}
\end{aligned}
$$

(a) Write down all elements of the sample space. [Hint: $\#S = 16$.]
(b) Compute the probability mass function for the sum $P(X + Y = k)$ and draw the probability histogram. [Hint: Count the outcomes corresponding to each value of k.]
(c) Compute the expected value $E[X + Y]$.
(d) Let $Z = \max\{X, Y\}$ be the maximum of the two numbers that show up. Compute the probability mass function $P(Z = k)$ and draw the probability histogram. [Hint: Count the outcomes corresponding to each value of k.]
(e) Compute the expected value $E[Z]$.

(a): If k shows up on the first roll and ℓ on the second roll then we will encode this outcome with the symbol "$k\ell$". Here is the sample space:

$$S = \{ \begin{array}{cccc} 11, & 12, & 13, & 14, \\ 21, & 22, & 23, & 24, \\ 31, & 32, & 33, & 34, \\ 41, & 42, & 43, & 44 \end{array} \}$$

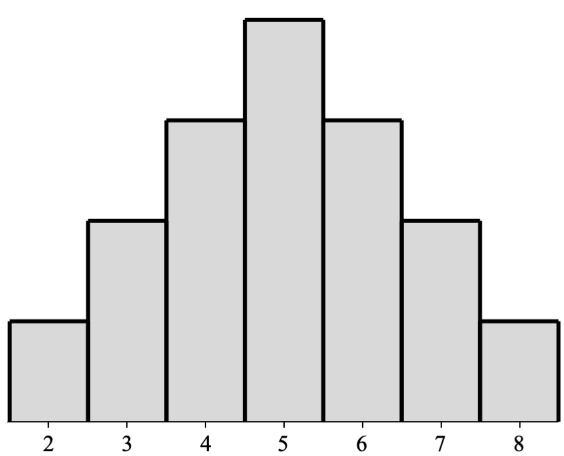
(b): We write down the outcomes corresponding to each possible value of $X + Y$:

$$\begin{aligned} \{X + Y = 2\} &= \{11\}, \\ \{X + Y = 3\} &= \{12, 21\} \\ \{X + Y = 4\} &= \{13, 22, 31\} \\ \{X + Y = 5\} &= \{14, 23, 32, 41\} \\ \{X + Y = 6\} &= \{24, 33, 42\} \\ \{X + Y = 7\} &= \{34, 43\} \\ \{X + Y = 8\} &= \{44\} \end{aligned}$$

Since the 16 elements of the sample space are equally likely, we obtain the following table:

k	2	3	4	5	6	7	8
$P(X + Y = k)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

And here is a histogram:



(c): There are three ways to compute the expected value of $X+Y$. (1) Symmetry: Since the distribution is symmetric about $X+Y=5$ we must have $E[X+Y]=5$. (2) Linearity: If we already know that $E[X]=E[Y]=2.5$ then we can use linearity to obtain $E[X+Y]=E[X]+E[Y]=2.5+2.5=5$. (3) Directly from the formula:

$$\begin{aligned} E[X+Y] &= \sum_k k \cdot P(X+Y=k) \\ &= 2\frac{1}{16}+3\frac{2}{16}+4\frac{3}{16}+5\frac{4}{16}+6\frac{3}{16}+7\frac{2}{16}+8\frac{1}{16}=5. \end{aligned}$$

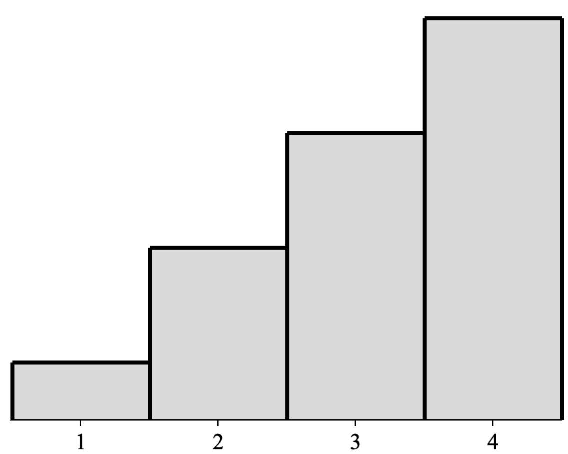
(d): We write down the outcomes corresponding to each possible value of $Z=\max\{X,Y\}$:

$$\begin{aligned} \{Z=1\} &= \{11\} \\ \{Z=2\} &= \{12,22,21\} \\ \{Z=3\} &= \{13,23,33,32,31\} \\ \{Z=4\} &= \{14,24,34,44,43,42,41\} \end{aligned}$$

Since the 16 elements of the sample space are equally likely, we obtain the following table:

k	1	2	3	4
$P(Z=k)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

And here is a histogram:



(e): The only way to compute $E[Z]$ is directly from the formula:

$$\begin{aligned} E[Z] &= \sum_k k \cdot P(Z=k) \\ &= 1\frac{1}{16}+2\frac{3}{16}+3\frac{5}{16}+4\frac{7}{16}=\frac{25}{8}=3.125 \end{aligned}$$

Remark: This is a testable prediction. Do the experiment and see if it works.

4. I am running a lottery. I will sell 100 tickets, each for a price of \$1. One of the tickets is a winner. The person who buys the winning ticket will receive a cash prize of \$90.

(a) Suppose you buy one ticket. If the ticket is a winner you will have a profit of \$89. If it is a loser you will have a profit of −\$1. What is the expected value of you profit?
(b) If you buy n tickets ($0 \le n \le 100$), what is the expected value of your profit?

(a): Let X be your expected profit from buying one ticket. The two possible values of X are $X = 89$ and $X = -1$. Since $X = 89$ corresponds to getting the winning ticket we have $P(X = 89) = 1/100$ and since $X = -1$ corresponds to **not** getting the winning ticket we have $P(X = -1) = 99/100$. Therefore the expected value is

$$E[X] = \sum_k kP(X = k) = 89P(X = 89) + (-1)P(X = -1) = 89\frac{1}{100} + (-1)\frac{99}{100} = -\frac{1}{10}.$$

In other words, if you buy one ticket then you should expect to lose ten cents.

(b): There are two ways to solve this.

The Good Way. Suppose you buy n tickets and let X_i be your expected profit from the ith ticket, so that $X = X_1 + X_2 + \cdots + X_n$. From part (a) we know that $E[X_i] = -1/10$ for all i. Thus from the linearity of expectation we have

$$\begin{aligned} E[X] &= E[X_1 + X_2 + \cdots + X_n] \\ &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= \underbrace{-1/10 - 1/10 - \cdots - 1/10}_{n \text{ times}} = -n/10. \end{aligned}$$

In other words, if you buy n tickets (with $0 \leq n \leq 100$) then you expect to lose $n/10$ dollars. For example, if you buy $n = 0$ tickets then you expect to lose 0 dollars, and if you buy $n = 100$ tickets then you expect to lose 10 dollars. Indeed, in this case you will pay me \$100 and I will give you back \$90 because you definitely won the prize.

The Bad Way. Suppose you buy n tickets and let X be your expected profit. There are only two possible values of X: If none of your tickets is a winner then $X = -n$ because of the cost of the tickets. And if one of your tickets is a winner then $X = 90 - n$ because you also gain the \$90. We just need to find the probabilities of these two outcomes. Think of an urn containing 100 tickets, 1 of which is a winner and 99 of which are not winners. If you grab n tickets then the probability of getting the winning ticket (hence $n - 1$ losing tickets) is

$$\begin{aligned} \frac{\binom{1}{1}\binom{99}{n-1}}{\binom{100}{n}} &= \frac{99!}{(n-1)!(99-n+1)!} \cdot \frac{n!(100-n)!}{100!} \\ &= \frac{n!}{(n-1)!} \cdot \frac{99!}{100!} \cdot \frac{(100-n)!}{(100-n)!} = \frac{n}{100}. \end{aligned}$$

Alternatively, let A_i be the event that your ith ticket is a winner. Since $P(A_i) = 1/100$ and since these events are mutually exclusive (two tickets cannot both be winners) we have

$$P(\text{one of your tickets is a winner}) = P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} \frac{1}{100} = \frac{n}{100}.$$

Next, the probability getting zero winning tickets (hence n losing tickets) is

$$\begin{aligned} \frac{\binom{1}{0}\binom{99}{n}}{\binom{100}{n}} &= \frac{99!}{n!(99-n)!} \cdot \frac{n!(100-n)!}{100!} \\ &= \frac{n!}{n!} \cdot \frac{99!}{100!} \cdot \frac{(100-n)!}{(100-n-1)!} = \frac{100-n}{100}. \end{aligned}$$

Alternatively, since zero and one winning tickets are the only two possibilities, we have

$$P(\text{zero winning tickets}) = 1 - P(\text{one winning ticket}) = 1 - \frac{n}{100} = \frac{100-n}{100}.$$

Finally, your expected profit is

$$\begin{aligned} E[X] &= \sum_k kP(X=k) \\ &= (-n)P(X=-n) + (90-n)P(X=90-n) \\ &= -n \cdot \frac{100-n}{100} + (90-n)\cdot\frac{n}{100} \\ &= \frac{-100n + n^2 + 90n - n^2}{100} = \frac{-10n}{100} = -\frac{n}{10}. \end{aligned}$$

Which method do you like better?

Remark: For this problem we did not have a ready made formula. We had to be a bit creative.

5. Let X be a geometric random variable with pmf

$$P(X=k) = pq^{k-1}.$$

(a) Use a geometric series to find a formula for $P(X > k)$.
(b) Use part (a) to find a formula for the *cumulative mass function* (cmf) $P(X \le k)$.
(c) Use part (b) to find a formula for the probability that X is between integers k and ℓ:

$$P(k \le X \le \ell) = ?$$

(a): Let's ignore the case $q = 1$. If $q < 1$ then we have

$$\begin{aligned} P(X > k) &= P(X=k+1) + P(X=k+2) + P(X=k+3) + P(X=k+4) + \cdots \\ &= pq^k + pq^{k+1} + pq^{k+2} + pq^{k+3} + \cdots \\ &= pq^k(1 + q^2 + q^3 + \cdots) \\ &= pq^k \frac{1}{1-q} = pq^k \frac{1}{p} = q^k. \end{aligned}$$

(b): It follows that

$$P(X \le k) = 1 - P(X > k) = 1 - q^k.$$

(c): And also[1]

$$P(k \le X \le \ell) = P(X > k-1) - P(X > \ell) = q^{k-1} - q^{\ell}.$$

6. The Coupon Collector Problem. Each box of a certain brand of cereal contains a coupon, selected at random from n different types of coupons. How many boxes will you need to purchase, on average, until you get all n types?

(a) Assume that you already have m types of coupons and let X_m be the number of boxes that you purchase until you get a type that you don't already have. Compute $E[X_m]$. [Hint: Think of each new box as a coin flip with $H =$"you get a new type of coupon" and $T =$"you get a coupon that you already have". Then X_m is a geometric random variable. What is the probability of H?]

[1] To sum over the values of X between k and ℓ, inclusive, we can sum over all the values greater than $k-1$ and then get rid of the values that are greater than ℓ.

(b) Let X be the number of boxes that you purchase until you get all n types of coupons. In the notation of part (a) we can write

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Use part (a) and linearity of expected value to compute $E[X]$.

(c) Example: Suppose you continue to roll a fair six-sided die until you see all six sides. On average, how many rolls do you expect to make?

(a): If we already have m types of coupons, then there are $n - m$ more that we still need to get. The chance of getting a new type in the next box is $(n - m)/n$. Until we get a new type, we can think of each new box as a coin flip with $P(H) = (n - m)/n$. Thus the number of boxes X_m until we get a new type is a geometric random variable with

$$E[X_m] = \frac{1}{P(H)} = \frac{1}{(n-m)/n} = \frac{n}{n-m}.$$

(b): The total number of boxes until we get all n types of coupons can be expressed as

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Hence from part (a) and the linearity of expectation we have

$$\begin{aligned} E[X] &= E[X_0 + X_1 + X_2 + \cdots + X_{n-1}] \\ &= E[X_0] + E[X_1] + E[X_2] + \cdots + E[X_{n-1}] \\ &= \frac{n}{n-0} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}. \end{aligned}$$

(c): For example, when $n = 6$ we have

$$E[X] = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 14.7.$$

Interpretation: On average, it takes 14.7 rolls to see all the faces of a fair six-sided die.

Remark: That problem is quite tricky, which is why it has a name. I would not have expected you to solve it without a big hint.

7. Expected Value of a Binomial. Let X be a binomial random variable with pmf

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

(a) For $n, k \geq 1$, use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(b) Use part (a) to compute the expected value of X. I'll get you started:

$$\begin{aligned} E[X] &= \sum_{k=0}^{n} kP(X = k) \\ &= \sum_{k=0}^{n} k\binom{n}{k} p^k q^{n-k} \\ &= \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} && \text{the } k = 0 \text{ term is zero} \\ &= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k q^{n-k} && \text{from part (a)} \end{aligned}$$

$$= \text{ now what?}$$

[Hint: Apply the binomial theorem to $(p+q)^{n-1}$.]

(a): We have

$$k\binom{n}{k} = \frac{k}{k!}\frac{n!}{(n-k)!} = \frac{1}{(k-1)!}\frac{n(n-1)!}{(n-k)!} = n\frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} = n\binom{n-1}{k-1}.$$

Alternatively, here is a counting proof: From a group of n people we will choose a committee of k people, one of whom will be the president of the committee. There are two ways to do this. On the one hand, we could first choose the committee in $\binom{n}{k}$ ways and then choose the president in k ways from the committee members. On the other hand, we could first choose the president in n ways and then choose the remaining committee members in $\binom{n-1}{k-1}$ ways.

(b): First we use the binomial theorem to observe that

$$1 = (p+q)^{n-1} = \sum_{\ell=0}^{n-1}\binom{n-1}{\ell}p^\ell q^{(n-1)-\ell}.$$

Then we can use this fact to complete the calculation:

$$\begin{aligned}
E[X] &= \sum_{k=0}^{n} kP(X=k) \\
&= \sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k} \\
&= \sum_{k=1}^{n} k\binom{n}{k}p^k q^{n-k} && \text{the } k=0 \text{ term is zero} \\
&= \sum_{k=1}^{n} n\binom{n-1}{k-1}p^k q^{n-k} && \text{from part (a)} \\
&= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}q^{n-k)} && \text{factor out } np \\
&= np\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}p^\ell q^{(n-1)-\ell} && \text{substitute } \ell = k-1 \\
&= np(p+q)^{n-1} && \text{from the observation} \\
&= np.
\end{aligned}$$

Remark: This was the bad way to do it. The good way is to express X as a sum of n Bernoulli random variables and use the linearity of expectation.