Math 224 **Fall 2021**
Homework 3 **Drew Armstrong**

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1. Function of a Random Variable. Let $X : S \to \mathbb{R}$ be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be an ordinary function. Then the composition $g(X)$ [do X first, then do g] is another random variable and we have the following formula:

$$E[g(X)] = \sum_{k \in S_X} g(k) \cdot P(X = k).$$

Now suppose that X is the number of heads obtained in two flips of a fair coin. Use this formula to compute the following expected values:

(a) $E[X + 1]$
(b) $E[X^2]$
(c) $E[2^X]$

2. An urn contains 3 red balls and 4 green balls. Suppose you grab 3 balls without replacement and let R be the number of red balls that you get.

(a) Find a formula for the pmf $P(R = k)$ and draw the probability histogram.
(b) Compute the expected value $E[R]$.

3. A fair four-sided die has sides labeled $\{1, 2, 3, 4\}$. Suppose you roll the die twice and consider the following random variables:

$$X = \text{the number that shows up on the first roll,}$$
$$Y = \text{the number that shows up on the second roll.}$$

(a) Write down all elements of the sample space. [Hint: $\#S = 16$.]
(b) Compute the probability mass function for the sum $P(X + Y = k)$ and draw the probability histogram. [Hint: Count the outcomes corresponding to each value of k.]
(c) Compute the expected value $E[X + Y]$.
(d) Let $Z = \max\{X, Y\}$ be the maximum of the two numbers that show up. Compute the probability mass function $P(Z = k)$ and draw the probability histogram. [Hint: Count the outcomes corresponding to each value of k.]
(e) Compute the expected value $E[Z]$.

4. I am running a lottery. I will sell 100 tickets, each for a price of \$1. One of the tickets is a winner. The person who buys the winning ticket will receive a cash prize of \$90.

(a) Suppose you buy one ticket. If the ticket is a winner you will have a profit of \$89. If it is a loser you will have a profit of $-\$1$. What is the expected value of you profit?
(b) If you buy n tickets ($0 \leq n \leq 100$), what is the expected value of your profit?

5. Let X be a geometric random variable with pmf

$$P(X = k) = pq^{k-1}.$$

(a) Use a geometric series to find a formula for $P(X > k)$.
(b) Use part (a) to find a formula for the *cumulative mass function* (cmf) $P(X \leq k)$.
(c) Use part (b) to find a formula for the probability that X is between integers k and ℓ:

$$P(k \leq X \leq \ell) = \,?$$

6. The Coupon Collector Problem. Each box of a certain brand of cereal contains a coupon, selected at random from n different types of coupons. How many boxes will you need to purchase, on average, until you get all n types?

(a) Assume that you already have m types of coupons and let X_m be the number of boxes that you purchase until you get a type that you don't already have. Compute $E[X_m]$. [Hint: Think of each new box as a coin flip with $H =$"you get a new type of coupon" and $T =$"you get a coupon that you already have". Then X_m is a geometric random variable. What is the probability of H?]

(b) Let X be the number of boxes that you purchase until you get all n types of coupons. In the notation of part (a) we can write

$$X = X_0 + X_1 + X_2 + \cdots + X_{n-1}.$$

Use part (a) and linearity of expected value to compute $E[X]$.

(c) Example: Suppose you continue to roll a fair six-sided die until you see all six sides. On average, how many rolls do you expect to make?

7. Expected Value of a Binomial. Let X be a binomial random variable with pmf

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

(a) For $n, k \geq 1$, use the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to show that $k\binom{n}{k} = n\binom{n-1}{k-1}$.

(b) Use part (a) to compute the expected value of X. I'll get you started:

$$\begin{aligned}
E[X] &= \sum_{k=0}^{n} kP(X = k) \\
&= \sum_{k=0}^{n} k\binom{n}{k} p^k q^{n-k} \\
&= \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} && \text{the } k = 0 \text{ term is zero} \\
&= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k q^{n-k} && \text{from part (a)} \\
&= \text{ now what?}
\end{aligned}$$

[Hint: Apply the binomial theorem to $(p + q)^{n-1}$.]