Math 224 **Fall 2021**
Homework 2 **Drew Armstrong**

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1. A Florida license plate consists of six characters: **four letters followed by two numbers**. Characters are allowed to be repeated.

(a) Find the number of possible license plates.
(b) If a license plate is chosen at random, what is the probability that it contains at least one vowel $\{A, E, I, O, U\}$? [Hint: What if it contains **no** vowels?]

(a): Since the characters are ordered we use the multiplication principle. Since characters are allowed to be repeated, the number of possibilities is

$$\underbrace{26}_{\text{1st letter}} \times \underbrace{26}_{\text{2nd letter}} \times \underbrace{26}_{\text{3rd letter}} \times \underbrace{26}_{\text{4th letter}} \times \underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{2nd digit}} = 45697600.$$

(b): We will assume that all license plates are equally likely. Instead of counting the license plates that contain at least one vowel, it is much easier to count the license plates that **do not have any vowels**:

$$\underbrace{21}_{\text{1st letter}} \times \underbrace{21}_{\text{2nd letter}} \times \underbrace{21}_{\text{3rd letter}} \times \underbrace{21}_{\text{4th letter}} \times \underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{2nd digit}} = 19448100.$$

We conclude that

$$\begin{aligned} P(\text{at least one vowel}) &= 1 - P(\text{no vowels}) \\ &= 1 - \frac{19448100}{45697600} \\ &= 57.4\%. \end{aligned}$$

Does this answer seem correct to you? I encourage you to go to the parking lot and test it out!

2. Suppose that a fair six-sided die has 3 sides painted red, 2 sides painted blue and 1 side painted green. Suppose you roll the die $n = 4$ times and let R, G, B be the number of times that you get red, green, blue, respectively.

(a) Compute $P(R = 1, G = 1, B = 2)$. [Hint: How many ways can it happen?]
(b) Compute $P(R \geq 1)$. [Hint: Think of the die as a coin.]
(c) Compute $P(G = B)$. [Hint: What are the possible values of R, G, B in this case?]

(a): We can think of the fair six-sided die as a "biased 3-sided die" with $P(R) = 3/6 = 1/2$, $P(G) = 1/6$ and $P(B) = 2/6 = 1/3$. Using the formula for multinomial probability gives

$$\begin{aligned} P(R = 1, G = 1, B = 2) &= \frac{4!}{1!1!2!} P(R)^1 P(G)^1 P(B)^2 \\ &= 12 \left(\frac{1}{2}\right)^1 \left(\frac{1}{6}\right)^1 \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{9} \text{ or } 11.1\% \end{aligned}$$

Remark: The formula $4!/(1!1!2!) = 12$ counts the number of ways to get $R = 1$, $G = 1$, $B = 2$. Here they are:

$$\begin{array}{cccccc} BBRG & BRBG & BRGB & RBBG & RBGB & RGBB \\ BBGR & BGBR & BGRB & GBBR & GBRB & GRBB \end{array}$$

(b): We can think of the fair six-sided die as a coin where heads is "red" and tails is "not red", so that $P(H) = 1/2$ and $P(T) = 1/2$. (It turns out that it is a fair coin.) If we flip the coin 4 times than

$$\begin{aligned} P(R \geq 1) &= 1 - P(R = 0) \\ &= 1 - P(\text{all non-red}) \\ &= 1 - P(T)^4 \\ &= 1 - (1/2)^4 \\ &= \frac{15}{16} \quad \text{or } 93.75\% \end{aligned}$$

(c): We will again think of the experiment as a 3-sided die. It turns out that there are exactly three outcomes corresponding to $G = B$. We obtain the probability of $G = B$ by adding them up (unfortunately, there is no quicker way to do this):

$$\begin{aligned} &P(G = B) \\ &= P(R = 4, G = 0, B = 0) + P(R = 2, G = 1, B = 1) + P(R = 0, G = 2, B = 2) \\ &= \frac{4!}{4!0!0!}\left(\frac{1}{2}\right)^4\left(\frac{1}{6}\right)^0\left(\frac{1}{3}\right)^0 + \frac{4!}{2!1!1!}\left(\frac{1}{2}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{3}\right)^1 + \frac{4!}{0!2!2!}\left(\frac{1}{2}\right)^0\left(\frac{1}{6}\right)^2\left(\frac{1}{3}\right)^2 \\ &= \frac{1}{16} + \frac{1}{6} + \frac{1}{54} \\ &= \frac{107}{432} \quad \text{or } 24.8\% \end{aligned}$$

3. The Birthday Problem. Consider a classroom of r students. Each student has a birthday, which we can encode as a number from the set $\{1, 2, \ldots, 365\}$ (ignore leap years). Assume that each birthday is equally likely.

(a) Suppose that the r students are ordered (for example, alphabetically by last name). If we record each student's birthday, what is the size of the sample space?
(b) Compute the probability of the event $E =$"some pair of students have the same birthday". [Hint: Consider the opposite $E' =$"no two students have the same birthday".]
(c) Find the smallest number of students r such that $P(E) > 50\%$. [Use a computer.]

(a) Using the multiplication principle gives

$$\#S = \underbrace{365}_{\substack{\text{1st student's} \\ \text{birthday}}} \times \underbrace{365}_{\substack{\text{2nd student's} \\ \text{birthday}}} \times \cdots \times \underbrace{365}_{\substack{r\text{th student's} \\ \text{birthday}}} = 365^r.$$

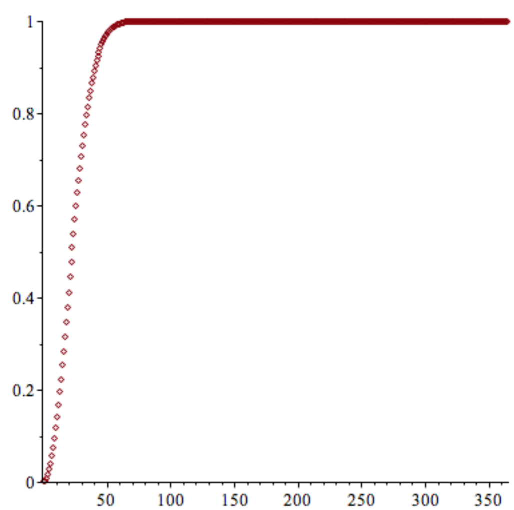
(b) If no two students are allowed to have the same birthday then for $r \geq 366$ we have $\#E' = 0$ and for $r \leq 365$ we have

$$\#E' = \underbrace{365}_{\substack{\text{1st student's}\\\text{birthday}}} \times \underbrace{364}_{\substack{\text{2nd student's}\\\text{birthday}}} \times \cdots \times \underbrace{(365 - r + 1)}_{\substack{r\text{th student's}\\\text{birthday}}} = \frac{365!}{(365-r)!}.$$

(c) If $r \geq 366$ we have $P(E') = 0$ and hence $P(E) = 1 - P(E') = 1$.[1] If $r \leq 365$ then assuming all birthdays are equally likely gives

$$\begin{aligned} P(\text{at least two share a birthday}) &= 1 - P(\text{no two share a birthday}) \\ P(E) &= 1 - P(E') \\ &= 1 - \frac{\#E'}{\#S} \\ &= 1 - \frac{365!/(365-r)!}{365^r}. \end{aligned}$$

(d) Here is a plot of the probabilites $P(E)$ for values of r from 1 to 365. Note that the probability rises from 0% when $r = 1$ to 100% when $r = 366$.



At some point the probability must cross 50% and it seems from the diagram that this happens around $r = 25$. To be precise, I used my computer to find the following:

- For $n = 22$ students, the probability that at least two share a birthday is
$$P(E) = 1 - \frac{365!/(365-22)!}{365^{22}} = 47.57\%.$$
- For $n = 23$ students, the probability that at least two share a birthday is
$$P(E) = 1 - \frac{365!/(365-23)!}{365^{23}} = 50.73\%.$$

Do you find the number 23 surprisingly small? That's why this problem is sometimes also called the **birthday paradox**.[2]

[1] Indeed, if there are at least 366 students then there are not enough days in the year for each student to have their own birthday. So at least two students must share.

[2] In our class of $r = 32$ students there is a 75.3% chance that two students share a birthday.

4. A Quick and Bad Proof of the Binomial Theorem.

(a) For all integers $r \geq 1$ show that $r! = r \times (r-1)!$. [Don't think too much.]

(b) For all integers $0 < k < n$, prove that

$$\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} = \frac{n!}{k!(n-k)!}.$$

[Hint: Use part (a) to get a common denominator.]

Part (a) is self-explanatory. For part (b) we use part (a) to get a common denominator:

$$\begin{aligned}
&\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{n-k}{n-k} \cdot \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k(n-1)!}{k(k-1)!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)(n-k-1)!} \\
&= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{[\cancel{k} + (n-\cancel{k})] \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k)!}.
\end{aligned}$$

5. Suppose that 5 cards are dealt at random and without replacement from a standard deck of 52 cards.[3] Find the probabilities of the following events:

(a) 1 club, 1 diamond, 2 hearts, 1 spade

(b) 1 club, 1 spade, 3 red cards

(c) 2 black cards, 3 red cards

[Hint: The analysis is easier if you assume that the cards are not ordered, so the size of the sample space is "52 choose 5" = 2598960.]

(a): The number of ways to choose 1 club, 1 diamond, 2 hearts and 1 spade is

$$\underbrace{\binom{13}{1}}_{\text{choose clubs}} \times \underbrace{\binom{13}{1}}_{\text{choose diamonds}} \times \underbrace{\binom{13}{2}}_{\text{choose hearts}} \times \underbrace{\binom{13}{1}}_{\text{choose spades}} = 171366,$$

hence the probability is

$$P(\text{1 club, 1 diamond, 2 hearts, 1 spade}) = \frac{\binom{13}{1}\binom{13}{1}\binom{13}{2}\binom{13}{1}}{\binom{52}{5}} = \frac{171366}{2598960} = 6.6\%.$$

[3] Each of the cards is labeled by one of four "suits" (clubs, diamonds, hearts, spades) and one of 13 "ranks" (1,2,...,10,J,Q,K,A), for a total of $4 \times 13 = 52$ cards. Clubs and spades are "black cards"; diamonds and hearts are "red cards".

(b): The number of ways to choose 1 club, 1 spade and 3 red cards is

$$\underbrace{\binom{13}{1}}_{\substack{\text{choose}\\\text{clubs}}} \times \underbrace{\binom{13}{1}}_{\substack{\text{choose}\\\text{spades}}} \times \underbrace{\binom{26}{3}}_{\substack{\text{choose}\\\text{red cards}}} = 439400,$$

hence the probability is

$$P(\text{1 club, 1 spade, 3 red cards}) = \frac{\binom{13}{1}\binom{13}{1}\binom{26}{3}}{\binom{52}{5}} = \frac{439400}{2598960} = 16.9\%.$$

(c): The number of ways to choose 2 black cards and 3 red cards is

$$\underbrace{\binom{26}{2}}_{\substack{\text{choose}\\\text{black cards}}} \times \underbrace{\binom{26}{3}}_{\substack{\text{choose}\\\text{red cards}}} = 845000,$$

hence the probability is

$$P(\text{2 black cards, 3 red cards}) = \frac{\binom{26}{2}\binom{26}{3}}{\binom{52}{5}} = \frac{845000}{2598960} = 32.5\%.$$

6. Two cards are drawn from a standard deck of 52 and placed sided by side on a table. Consider the following events:

$$A = \text{"the left card is a heart"},$$
$$B = \text{"the right card is black"}.$$

Compute the following probabilities:

$$P(A), \quad P(B), \quad P(B|A), \quad P(A \cap B), \quad P(A|B).$$

Ignoring the right card gives $P(A) = 13/52 = 1/4$ and ignoring the left card gives $P(B) = 26/52 = 1/2$. If the left card is a heart then there are 51 remaining cards and 26 remaining black cards, so that

$$P(B|A) = P(\text{right card is black, assuming left card is a heart}) = 26/51.$$

Similarly, if the right card is black then there are 51 remaining cards and 13 remaining hearts, so that

$$P(B|A) = P(\text{left card is a heart, assuming that right card is black}) = 13/51.$$

Finally, we have a few different ways to compute $P(A \cap B)$:

- $P(A \cap B) = P(A)P(B|A) = (1/4)(26/51) = 13/102$
- $P(A \cap B) = P(B)P(AB) = (1/2)(13/51) = 13/102$
- Or we could just count. The number of ways to choose two ordered cards (call them the left card and the right card) is

$$\underbrace{52}_{\substack{\text{choose}\\\text{left card}}} \times \underbrace{51}_{\substack{\text{choose}\\\text{right card}}} = 2652.$$

And the number of ways to choose two ordered cards where the left card is a heart and the right card is black is

$$\underbrace{13}_{\substack{\text{choose}\\\text{heart}}} \times \underbrace{26}_{\substack{\text{choose}\\\text{black card}}} = 338.$$

Hence the probability is

$$P(\text{left is red and right is black}) = \frac{338}{2652} = 13/102.$$

7. Bayes' Theorem. There are two bowls on a table. The first bowl contains 2 red chips and 3 green chips. The second bowl contains 4 red chips and 2 green chips. Your friend walks up to the table, chooses a bowl at random, and then chooses a chip at random. Assume that the two bowls are equally likely, and after having chosen a bowl, assume that the chips in that bowl are equally likely. Consider the events:

$$\begin{aligned} B_1 &= \text{"the chip came from the first bowl"}, \\ B_2 &= \text{"the chip came from the second bowl"}, \\ R &= \text{"the chip is red"}. \end{aligned}$$

(a) Compute the forwards probabilities $P(R|B_1)$ and $P(R|B_2)$.
(b) Compute the probability $P(R)$ that the chip is red. [Hint: The answer is not $6/11$ because the 11 chips in the two bowls are **not** equally likely. Actually, each chip in bowl 1 is slightly more likely than each chip in bowl 2 because bowl 1 has fewer chips.]
(c) Compute the backwards probability $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that this chip came from the first bowl?

(a) We have $P(R|B_1) = \frac{2}{2+3} = \frac{2}{5}$ and $P(R|B_2) = \frac{4}{4+2} = \frac{2}{3}$.

(b) The Law of Total Probability gives

$$\begin{aligned} R &= (R \cap B_1) \cup (R \cap B_2) \\ P(R) &= P(R \cap B_1) + P(R \cap B_2) \\ P(R) &= P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) \\ &= \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{2}{3} = \frac{8}{15}. \end{aligned}$$

(c) Bayes' Theorem gives

$$P(B_1|R) = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{(1/2)(2/5)}{8/15} = \frac{3}{8} = 37.5\%.$$

[Remark: Before we know the color of the chip, there is a 50% chance that it came from the first bowl. After we know that the chip is red, there is a 37.5% chance that it came from the first bowl. There is a philosophical issue in the distinction between forwards and backwards probability. Forwards probability predicts the outcome of an experiment in the future. Backwards probability is a measure of our incomplete knowledge of the past. Or something like that. But the equations governing both are the same.]