Math 224 **Exam 3**
Fall 2021 **Tues Dec 7**

Problem 1. Let X be a continuous random variable with the following density function:

$$f_X(x) = \begin{cases} cx^2 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

The integral of the density must equal 1, so that

$$1 = \int_0^1 cx^2\,dx = c\left[\frac{x^3}{3}\right]_0^1 = c \cdot \frac{1}{3}.$$

It follows that $c = 3$.

(b) Compute the expected value $E[X]$.

From part (a) and the definition of expected value, we have

$$E[X] = \int_0^1 x \cdot 3x^2\,dx = 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}.$$

(c) Compute the second moment $E[X^2]$ and the variance $\mathrm{Var}(X)$.

Following from part (b), we have

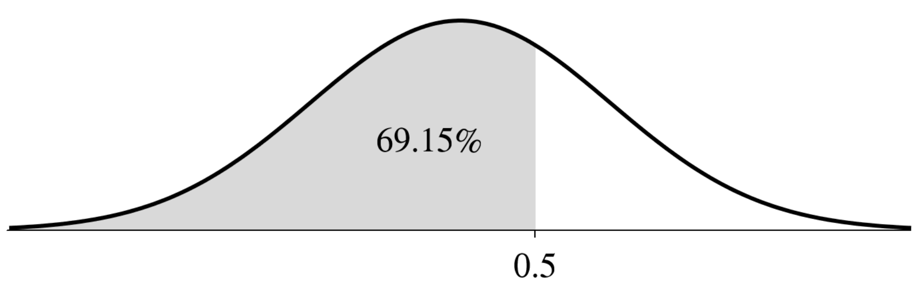
$$E[X^2] = \int_0^1 x^2 \cdot 3x^2\,dx = 3\int_0^1 x^4\,dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5},$$

and hence

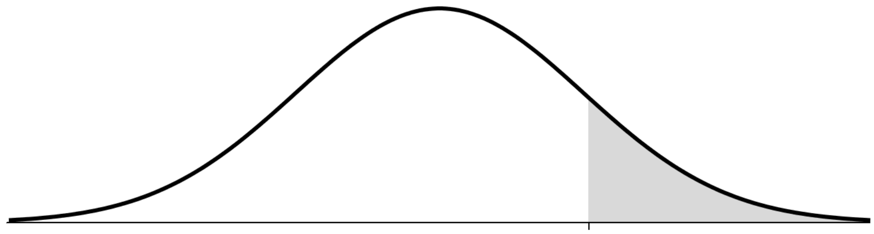
$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.375.$$

Problem 2. Let $Z \sim N(0,1)$ be a standard normal random variable. Use the attached tables to solve the following problems.

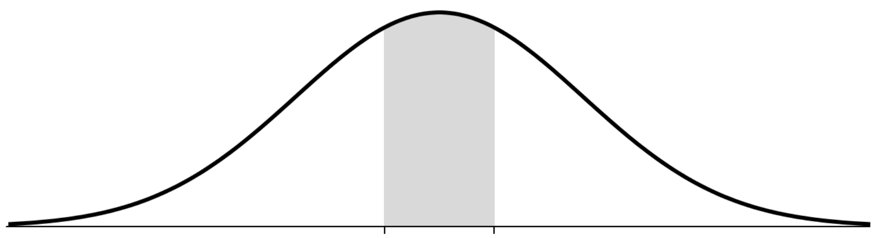
(a) Find α such that $P(Z < 0.5) = \alpha$ and draw a picture to illustrate your answer.



(b) Find a such that $P(Z > a) = 15\%$ and draw a picture to illustrate your answer.



(c) Find b such that $P(|Z| < b) = 30\%$ and draw a picture to illustrate your answer.



Problem 3. Let $X_1, \ldots, X_{25}$ be an iid sample from a distribution with mean $\mu = 5$ and variance $\sigma^2 = 1$. (The distribution might not be normal.) Consider the sample mean:

$$\overline{X} = \frac{1}{25}\left(X_1 + X_2 + \cdots + X_{25}\right).$$

Use the attached tables to solve the following problems.

(a) Compute the expected value of the sample mean, $E[\overline{X}]$.

We use linearity and the fact that $E[X_i] = 5$ for each i:

$$E[\overline{X}] = \frac{1}{25}\left(E[X_1] + \cdots + E[X_{25}]\right) = \frac{1}{25}(5 + \cdots + 5) = \frac{1}{25} \cdot 25 \cdot 5 = 5.$$

(b) Compute the variance of the sample mean, $\mathrm{Var}(\overline{X})$.

We use independence and the fact that $\mathrm{Var}(X_i) = 1$ for each i:

$$\begin{aligned}\mathrm{Var}(\overline{X}) &= \left(\frac{1}{25}\right)^2 \left(\mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_{25})\right) \\ &= \left(\frac{1}{25}\right)^2 (1 + \cdots + 1) = \left(\frac{1}{25}\right)^2 \cdot 25 = \frac{1}{25}.\end{aligned}$$

(c) Use the Central Limit Theorem to estimate the probability $P(4.9 < \overline{X} < 6.1)$.

The Central Limit Theorem tells us that $\overline{X}$ is approximately normal. Since $E[\overline{X}] = 5$ and $\mathrm{Var}(\overline{X}) = 1/25$, this implies that

$$\frac{\overline{X} - 5}{\sqrt{1/25}} = \frac{\overline{X} - 5}{1/5} = 5 \cdot \overline{X} - 25 \text{ is approximately standard normal.}$$

We use this to estimate the desired probability. OOPS, I meant to write $P(4.9 < \overline{X} < 5.1)$ instead of $P(4.9 < \overline{X} < 6.1)$. Let's do both:

$$\begin{aligned}
P\left(4.9 < \overline{X} < 5.1\right) &= P\left(5 \cdot 4.9 - 25 < 5 \cdot \overline{X} - 25 < 5 \cdot 5.1 - 25\right) \\
&= P(-0.5 < 5 \cdot \overline{X} - 25 < 0.5) \\
&\approx \Phi(0.5) - \Phi(-0.5) \\
&= \Phi(0.5) - [1 - \Phi(0.5)] \\
&= 2 \cdot \Phi(0.5) - 1 \\
&= 2(0.6915) - 1 \\
&= 38.3\%
\end{aligned}$$

$$\begin{aligned}
P\left(4.9 < \overline{X} < 6.1\right) &= P\left(5 \cdot 4.9 - 25 < 5 \cdot \overline{X} - 25 < 5 \cdot 6.1 - 25\right) \\
&= P(-0.5 < 5 \cdot \overline{X} - 25 < 5.5) \\
&\approx \Phi(5.5) - \Phi(-0.5) \\
&= \Phi(5.5) - [1 - \Phi(0.5)] \\
&= 1 - [1 - 0.6915] && \Phi(5.5) \approx 1 \\
&= 69.15\%.
\end{aligned}$$

Problem 4. Consider a coin with unknown probability of heads, $P(H) = p$. Suppose that you flip the coin $n = 200$ times and get heads $Y = 120$ times. Use the attached tables to solve the following problems.

(a) Compute a two-sided 95% confidence interval for the unknown constant p.

We will use the estimator $\hat{p} = Y/n = 120/200 = 0.6$. The easiest formula for a symmetric, two-sided, approximate $(1-\alpha)100\%$ confidence interval is

$$p = \hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

When $\alpha = 5\%$ we have $z_{\alpha/2} = z_{2.5\%} = 1.96$. Hence our confidence interval is

$$p = 0.6 \pm 1.96 \cdot \sqrt{\frac{0.6(1-0.6)}{200}} = 0.6 \pm 0.068.$$

In other words, we are 95% confident that the unknown constant p falls in the interval between $0.6 - 0.068 = 0.532$ and $0.6 + 0.068 = 0.668$.

(b) Test the hypothesis $H_0 =$ "$p = 1/2$" against the alternative $H_1 =$ "$p \neq 1/2$" at the 1% level of significance.

Again we use $\hat{p} = Y/n = 0.6$ as an estimate for p. With the null hypothesis $H_0 =$ "$p = p_0$" and alternative $H_1 =$ "$p \neq p_0$", the typical rejection region for a hypothesis test with significance α has the form

$$|\hat{p} - p_0| > z_{\alpha/2}\sqrt{\frac{p_0(1-p_0)}{n}}.$$

When $\alpha = 1\%$ we have $z_{\alpha/2} = z_{0.5\%} = 2.575$. Then $p_0 = 0.5$ and $n = 200$ give

$$|\hat{p} - 0.5| > 2.575\sqrt{\frac{0.5(1-0.5)}{200}} = 0.091.$$

Since $|\hat{p} - 0.5| = |0.6 - 0.5| = 0.1$ is greater than 0.091 we **reject** H_0 in favor of H_1.

Problem 5. Suppose that the following iid sample comes from a **normal distribution** with unknown mean μ and variance σ^2:

4.9	4.0	4.2	4.8

Use the attached tables to solve the following problems.

(a) Compute a two-sided 98% confidence interval for the unknown constant μ.

For a normal sample, the following is a symmetric two-sided $(1-\alpha)100\%$ confidence interval for the unknown mean μ:

$$\mu = \overline{X} \pm t_{\alpha/2}(n-1)\sqrt{\frac{S^2}{n}}.$$

In our case, the sample mean and sample variance are

$$\overline{X} = \frac{1}{4}(4.9 + 4.0 + 4.2 + 4.8) = \frac{1}{4} \cdot 17.9 = 4.475$$

and

$$\begin{aligned} S^2 &= \frac{1}{3}\left[(4.9-4.475)^2 + (4.0-4.475)^2 + (4.2-4.475)^2 + (4.8-4.475)^2\right] \\ &= \frac{1}{3}[0.180625 + 0.225625 + 0.075625 + 0.105625] \\ &= \frac{1}{3} \cdot 0.5875 \\ &= 0.195833. \end{aligned}$$

When $\alpha = 2\%$ we have $t_{\alpha/2}(3) = t_{1\%}(3) = 4.541$ and plugging everything in gives

$$\mu = 4.475 \pm 4.541\sqrt{\frac{0.195833}{4}} = 4.475 \pm 1.005.$$

In other words, we are 98% confident that μ is between 3.47 and 5.48.

(b) Test the hypothesis $H_0 =$ "$\mu = 5$" against the alternative $H_1 =$ "$\mu < 5$" at the 5% level of significance.

The rejection region for the test of "$\mu = \mu_0$" against "$\mu < \mu_0$" is

$$\overline{X} < \mu_0 - t_\alpha(n-1)\sqrt{\frac{S^2}{n}}.$$

When $\mu_0 = 5$ and $\alpha = 5\%$ this becomes

$$\overline{X} < 5 - t_{5\%}(3)\sqrt{\frac{0.195833}{4}} = 5 - 2.353\sqrt{\frac{0.195833}{4}} = 4.479.$$

Since $\overline{X} = 4.475$ is less than 4.479 we **reject** H_0 in favor of H_1. (But it's close!)