Math 224 **Exam 2**
Fall 2021 **Thurs Oct 28**

This is a closed book test. No electronic devices are allowed. There are 5 pages and 5 problems, each worth 6 points, for a total of 30 points.

Problem 1. An urn contains 2 red and 4 green balls. Suppose that you reach in and grab 2 balls without replacement, and let X be the number of red balls you get.

(a) Fill in the following pmf table:

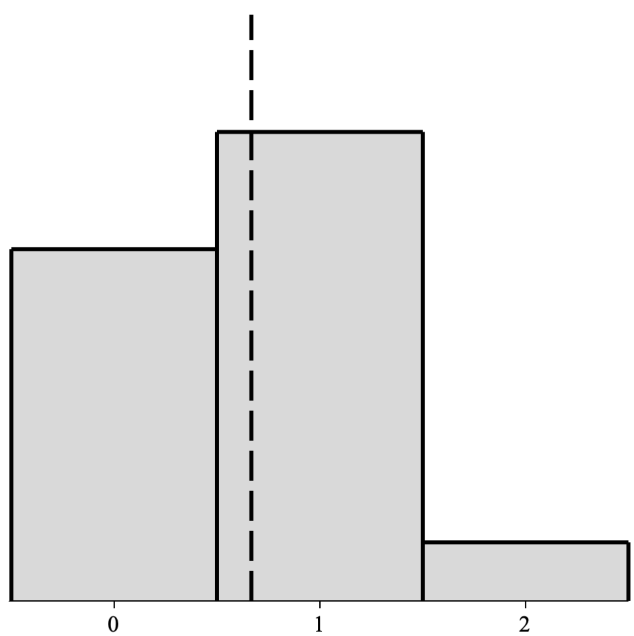
k	0	1	2
$P(X=k)$	$\frac{\binom{2}{0}\binom{4}{2}}{\binom{6}{2}} = \frac{6}{15}$	$\frac{\binom{2}{1}\binom{4}{1}}{\binom{6}{2}} = \frac{8}{15}$	$\frac{\binom{2}{2}\binom{4}{0}}{\binom{6}{2}} = \frac{1}{15}$

(b) Use your table to compute the expected value $E[X]$.

$$E[X] = \sum_k k \cdot P(X=k) = 0 \cdot \frac{6}{15} + 1 \cdot \frac{8}{15} + 2 \cdot \frac{1}{15} = \frac{10}{15} = \frac{2}{3}$$

Alternatively, we can use the formula $E[X] = nr/(r+g) = 2 \cdot 2/(2+4) = 4/6 = 2/34$.

(c) Draw a histogram, showing the pmf and the expected value of X.



Problem 2. Let $p \geq 0$ and $q \geq 0$ with $p + q = 1$. Let $X_1, X_2, \ldots, X_n$ be a sequence of *independent* random variables, each with the same pmf table:

k	0	1
$P(X_i = k)$	q	p

(a) Compute $E[X_i]$ and $\text{Var}(X_i)$.

$$\begin{aligned} E[X_i] &= 0 \cdot P(X_i = 0) + 1 \cdot P(X_i = 1) = 0q + 1p = p, \\ E[X_i^2] &= 0^2 \cdot P(X_i = 0) + 1^2 \cdot P(X_i = 1) = 0q + 1p = p, \\ \text{Var}(X_i) &= E[X_i^2] - E[X_i]^2 = p - p^2 = p(1-p) = pq. \end{aligned}$$

Alternatively, we can use the formulas for the expected value and variance of a Bernoulli random variable.

(b) Compute $E[X_1 + X_2 + \cdots + X_n]$.

From linearity of expectation we have

$$\begin{aligned} E[X_1 + X_2 + \cdots + X_n] &= E[X_1] + E[X_2] + \cdots + E[X_n] \\ &= \underbrace{p + p + \cdots + p}_{n \text{ times}} = np. \end{aligned}$$

(c) Compute $\text{Var}(X_1 + X_2 + \cdots + X_n)$.

Since the X_i are independent we have

$$\begin{aligned} \text{Var}(X_1 + X_2 + \cdots + X_n) &= \text{Var}(X_1) + \text{Var}(X_2) + \cdots + \text{Var}(X_n) \\ &= \underbrace{pq + pq + \cdots + pq}_{n \text{ times}} = npq. \end{aligned}$$

Alternatively, we can use the formulas for the expected value and variance of a binomial random variable.

Problem 3. Let X, Y be random variables with the following moments:

$$E[X] = 2, \quad E[X^2] = 5, \quad E[Y] = 2, \quad E[Y^2] = 6, \quad E[XY] = 7.$$

(a) Compute $E[(X+1)^2]$.

$$\begin{aligned} E[(X+1)^2] &= E[X^2 + 2X + 1] \\ &= E[X^2] + 2 \cdot E[X] + 1 \\ &= 5 + 2 \cdot 2 + 1 \\ &= 10 \end{aligned}$$

(b) Compute Var(X), Var(Y) and Cov(X, Y).

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 = 5 - 2^2 = 1, \\
\mathrm{Var}(Y) &= E[Y^2] - E[Y]^2 = 6 - 2^2 = 2, \\
\mathrm{Cov}(X, Y) &= E[XY] - E[X] \cdot E[Y] = 7 - 2 \cdot 2 = 3.
\end{aligned}$$

(c) Compute Cov$(X + Y, 2X + 3Y)$.

$$\begin{aligned}
\mathrm{Cov}(X + Y, 2X + 3Y) &= 2\mathrm{Cov}(X, X) + 3\mathrm{Cov}(X, Y) + 2\mathrm{Cov}(Y, X) + 3\mathrm{Cov}(Y, Y) \\
&= 2\mathrm{Var}(X) + 5\mathrm{Cov}(X, Y) + 3\mathrm{Var}(Y) \\
&= 2 \cdot 1 + 5 \cdot 3 + 3 \cdot 2 \\
&= 23
\end{aligned}$$

Problem 4. Let X, Y be discrete random variables with the following joint pmf table:

$X \setminus Y$	1	2
1	1/10	2/10
2	3/10	4/10

(a) Fill in the following tables:

k	1	2
$P(X = k)$	$\frac{3}{10}$	$\frac{7}{10}$

ℓ	1	2
$P(Y = \ell)$	$\frac{4}{10}$	$\frac{6}{10}$

(b) Compute the expected values $E[X]$ and $E[Y]$.

$$\begin{aligned}
E[X] &= 1 \cdot (3/10) + 2 \cdot (7/10) = 17/10 \\
E[Y] &= 1 \cdot (4/10) + 2 \cdot (6/10) = 16/10
\end{aligned}$$

(c) Compute the mixed moment $E[XY]$ and the covariance Cov(X, Y).

$$E[XY] = 1 \cdot 1 \cdot (1/10) + 1 \cdot 2 \cdot (2/10) + 2 \cdot 1 \cdot (3/10) + 2 \cdot 2 \cdot (4/10) = 27/10$$

$$\mathrm{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = (27/10) - (17/10) \cdot (16/10) = -1/50$$

Problem 5. Consider a fair six-sided die with sides labeled $\{1,2,3,4,5,6\}$. Roll the die *twice* and consider the following random variables:

$$\begin{aligned} X &= \text{the number of times you get "1"}, \\ Y &= \text{the number of times you get "2"}. \end{aligned}$$

(a) Compute $E[X]$ and $E[Y]$.

Note that X and Y are each binomial with $n = 2$ and $p = 1/6$, so that

$$E[X] = E[Y] = np = 2(1/6) = 2/6 = 1/3$$

(b) Compute $\text{Var}(X)$ and $\text{Var}(Y)$.

Using the formula $\text{Var}(X) = npq$ for a binomial random variable gives

$$\text{Var}(X) = \text{Var}(Y) = npq = 2(1/6)(5/6) = 10/36$$

(c) Compute $\text{Var}(X+Y)$ and use the result to find $\text{Cov}(X,Y)$. [Hint: $X+Y$ is a binomial random variable.]

Note that $X+Y$ is the number of times we get "1 or 2" in two rolls of a fair die. This is a binomial random variable with $n = 2$ and $p = 2/6$, hence

$$\text{Var}(X+Y) = npq = 2(2/6)(4/6) = 16/36.$$

Then it follows from part (b) that

$$\begin{aligned} \text{Var}(X) + \text{Var}(Y) + 2 \cdot \text{Cov}(X,Y) &= \text{Var}(X+Y) \\ 2 \cdot \text{Cov}(X,Y) &= \text{Var}(X+Y) - \text{Var}(X) - \text{Var}(Y) \\ 2 \cdot \text{Cov}(X,Y) &= 16/36 - 10/36 - 10/36 \\ 2 \cdot \text{Cov}(X,Y) &= -4/36 \\ \text{Cov}(X,Y) &= -2/36. \end{aligned}$$

Alternatively, we can quote the formula for multinomial covariance:

$$\text{Cov}(X_i, X_j) = -np_ip_j = -2(1/6)(1/6) = -2/36.$$

Some students chose not to follow the hint and instead used the joint pmf:

$$P(X = k, Y = \ell) = \frac{2!}{k!\ell!(2-k-\ell)!} \left(\frac{1}{6}\right)^k \left(\frac{1}{6}\right)^\ell \left(\frac{4}{6}\right)^{2-k-\ell}.$$

Then they used the table to compute $E[XY]$ and hence $\text{Cov}(X,Y)$:

$X \setminus Y$	0	1	2
0	16/36	8/36	1/36
1	8/36	2/36	0
2	1/36	0	0