**Math 224** **Fall 2017**
**Homework 2** **Drew Armstrong**

Problems from 9th edition of *Probability and Statistical Inference* by Hogg, Tanis and Zimmerman:

- Section 1.2, Exercises 5, 7, 13, 16.
- Section 1.3, Exercises 4, 6, 7, 11.
- Section 1.5, Exercises 2, 4.

**Solutions to Book Problems.**

**1.2-5.** How many four-letter code words are possible using the letters IOWA if

(a) The letters may not be repeated? *Answer:*

$$\underbrace{4}_{\text{1st letter}} \times \underbrace{3}_{\text{2nd letter}} \times \underbrace{2}_{\text{3rd letter}} \times \underbrace{1}_{\text{4th letter}} = 4! = 24.$$

(b) The letters may be repeated? *Answer:*

$$\underbrace{4}_{\text{1st letter}} \times \underbrace{4}_{\text{2nd letter}} \times \underbrace{4}_{\text{3rd letter}} \times \underbrace{4}_{\text{4th letter}} = 4^4 = 256.$$

**1.2-7.** In a state lottery, four digits are drawn (one at a time and with replacement) from the possibilities $0, 1, 2, \ldots, 9$. Let $S$ be the sample space of all possible outcomes, so that

$$\#S = \underbrace{10}_{\text{1st digit}} \times \underbrace{10}_{\text{2nd digit}} \times \underbrace{10}_{\text{3rd digit}} \times \underbrace{10}_{\text{4th digit}} = 10^4 = 10,000.$$

Suppose that you win if any permutation of your selected integers is drawn. What is the probability of winning if you select

(a) $6, 7, 8, 9$. *Answer:* The number of permutations of $6, 7, 8, 9$ is

$$\binom{4}{1,1,1,1} = \frac{4!}{1!1!1!1!} = 24,$$

so the probability of winning is

$$P(\text{winning}) = \frac{24}{10,000} = 0.24\%.$$

(b) $6, 7, 8, 8$. *Answer:* The number of permutations of $6, 7, 8, 8$ is

$$\binom{4}{1,1,2} = \frac{4!}{1!1!2!} = 12,$$

so the probability of winning is

$$P(\text{winning}) = \frac{12}{10,000} = 0.12\%.$$

(c) $7, 7, 8, 8$. *Answer:* The number of permutations of $7, 7, 8, 8$ is

$$\binom{4}{2,2} = \frac{4!}{2!2!} = 6,$$

so the probability of winning is

$$P(\text{winning}) = \frac{6}{10,000} = 0.06\%.$$

(d) $7, 8, 8, 8$. *Answer:* The number of permutations of $7, 8, 8, 8$ is

$$\binom{4}{1,3} = \frac{4!}{1!3!} = 4,$$

so the probability of winning is

$$P(\text{winning}) = \frac{4}{10,000} = 0.04\%.$$

**1.2-13.** A bridge hand consists of 13 (unordered) cards taken (at random and without replacement) from a standard deck of 52 cards. Let $S$ be the sample space of all possible bridge hands, so that

$$\#S = \binom{52}{13} = \frac{52!}{13!39!} = 635,013,559,600.$$

Find the probability of each of the following hands.

(a) 5 spades, 4 hearts, 3 diamonds, 1 club. *Answer:* The number of such hands is

$$\underbrace{\binom{13}{5}}_{\text{choose spades}} \times \underbrace{\binom{13}{4}}_{\text{choose hearts}} \times \underbrace{\binom{13}{3}}_{\text{choose diamonds}} \times \underbrace{\binom{13}{1}}_{\text{choose clubs}} = 3,421,322,190$$

so the probability of this hand is

$$\frac{3,421,322,190}{635,013,559,600} \approx 0.54\%.$$

(b) 5 spades, 4 hearts, 2 diamonds, 2 clubs. *Answer:* The number of such hands is

$$\underbrace{\binom{13}{5}}_{\text{choose spades}} \times \underbrace{\binom{13}{4}}_{\text{choose hearts}} \times \underbrace{\binom{13}{2}}_{\text{choose diamonds}} \times \underbrace{\binom{13}{2}}_{\text{choose clubs}} = 5,598,527,220$$

so the probability of this hand is

$$\frac{5,598,527,220}{635,013,559,600} \approx 0.88\%.$$

(c) 5 spades, 4 hearts, 1 diamond, 3 clubs. *Answer:* The number of such hands is

$$\underbrace{\binom{13}{5}}_{\text{choose spades}} \times \underbrace{\binom{13}{4}}_{\text{choose hearts}} \times \underbrace{\binom{13}{1}}_{\text{choose diamonds}} \times \underbrace{\binom{13}{3}}_{\text{choose clubs}} = 3,421,322,190$$

so the probability of this hand is

$$\frac{3,421,322,190}{635,013,559,600} \approx 0.54\%.$$

(d) Suppose you are dealt 5 cards of one suit (say spades) and 4 cards of another suit (say hearts). Is it more likely that the other suits split $2, 2$ or split $1, 3$? *Answer:* There are 4 cards remaining to be dealt from the two remaining suits (in this example, diamonds and clubs). If the cards split $2,2$ then we must have 2 diamonds and 2 clubs. The number of ways to do this is

$$\underbrace{\binom{13}{2}}_{\substack{\text{choose}\\\text{diamonds}}} \times \underbrace{\binom{13}{2}}_{\substack{\text{choose}\\\text{clubs}}} = 6,084.$$

If the cards split $1,3$ then we might have 1 diamond and 3 clubs or we might have 3 diamonds and 1 club. Thus the total number of possibilities is

$$\underbrace{\binom{13}{1}}_{\substack{\text{choose}\\\text{diamonds}}} \times \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\\text{clubs}}} + \underbrace{\binom{13}{3}}_{\substack{\text{choose}\\\text{diamonds}}} \times \underbrace{\binom{13}{1}}_{\substack{\text{choose}\\\text{clubs}}} = 7,436.$$

We conclude that splitting $1,3$ is more likely than splitting $2,2$.

**1.2-16.** A box of candy hearts contains 52 hearts, of which 19 are white, 10 are tan, 7 are pink, 3 are purple, 5 are yellow, 2 are orange, and 6 are green. Suppose you select 9 (unordered) pieces of candy (randomly and without replacement) from the box. Let $S$ be the sample space so that

$$\#S = \binom{52}{9} = 3,679,075,400.$$

Give the probability that

(a) Three of the hearts are white. *Answer:* The number of choices is

$$\underbrace{\binom{19}{3}}_{\substack{\text{choose white}\\\text{hearts}}} \times \underbrace{\binom{33}{6}}_{\substack{\text{choose non-white}\\\text{hearts}}} = 1,073,233,392$$

so the probability is

$$\frac{1,073,233,392}{3,679,075,400} \approx 29.17\%.$$

(b) 3 white, 2 tan, 1 pink, 1 yellow, 2 green. *Answer:* The number of choices is

$$\underbrace{\binom{19}{3}}_{\text{white}} \times \underbrace{\binom{10}{2}}_{\text{tan}} \times \underbrace{\binom{7}{1}}_{\text{pink}} \times \underbrace{\binom{5}{1}}_{\text{yellow}} \times \underbrace{\binom{6}{2}}_{\text{green}} = 22,892,625$$

so the probability is

$$\frac{22,892,625}{3,679,075,400} \approx 0.622\%.$$

**1.3-4.** Two cards are drawn (successively and without replacement) from a standard deck of 52 cards. If $S$ is the sample space then we have

$$\#S = \underbrace{52}_{\text{1st card}} \times \underbrace{51}_{\text{2nd card}} = 2,652.$$

Compute the probability of drawing

(a) Two hearts. *Answer:* The number of choices is

$$\underbrace{13}_{\text{heart}} \times \underbrace{12}_{\text{heart}} = 156$$

so the probability is

$$P(\text{two hearts}) = \frac{13 \times 12}{52 \times 51} \approx 5.88\%.$$

(b) 1st draw heart, 2nd draw club. *Answer:* The number of choices is

$$\underbrace{13}_{\text{heart}} \times \underbrace{13}_{\text{club}} = 169$$

so the probability is

$$P(\text{1st heart, 2nd club}) = \frac{13 \times 13}{52 \times 51} \approx 6.37\%.$$

(c) 1st draw heart, 2nd draw ace. *Answer:* To count these we need to isolate the ace of hearts. The number of choices is

$$\underbrace{12}_{\text{heart}} \times \underbrace{1}_{\substack{\text{ace of}\\ \text{hearts}}} + \underbrace{13}_{\text{heart}} \times \underbrace{3}_{\substack{\text{ace of}\\ \text{non-hearts}}} = 51$$

so the probability is

$$P(\text{1st heart, 2nd ace}) = \frac{12 \times 1 + 13 \times 3}{52 \times 51} \approx 1.92\%.$$

**1.3-6.** A man is selected at random from a group of 982 men who died in 2002. Consider the events

$A =$ "the man died from heart disease,"

$B =$ "the man had at least one parent who had some heart disease."

We are told that

$$P(A) = \frac{221}{982}, \quad P(B) = \frac{334}{982} \quad \text{and} \quad P(A \cap B) = \frac{111}{982}.$$

Given that neither of his parents had heart disease, find the conditional probability that this man died from heart disease.

*Solution:* We are looking for the probability $P(A|B')$, which by definition is

$$P(A|B') = \frac{P(A \cap B')}{P(B')}.$$

We know that $P(B') = 1 - P(B)$ so it remains only to compute $P(A \cap B')$. To do this we can use $B$ to divide $A$ into two disjoint pieces:

$$\begin{aligned} A &= (A \cap B) \sqcup (A \cap B') \\ P(A) &= P(A \cap B) + P(A \cap B') \\ P(A) - P(A \cap B) &= P(A \cap B'). \end{aligned}$$

Finally, we conclude that

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{221 - 111}{982 - 334} \approx 16.98\%.$$

**1.3-7.** An urn contains 2 orange and 2 blue balls. Your friend selects 2 balls (at random and without replacement) and tells you that at least one of them is orange. What is the probability that the other ball is also orange?

*Solution:* The sample space satisfies $\#S = \binom{4}{2} = 6$. Let $X$ be the number of orange balls in your friend's selection so that

$$P(X=0) = \frac{\binom{2}{0}\binom{2}{2}}{\binom{4}{2}} = \frac{1}{6}, \quad P(X=1) = \frac{\binom{2}{1}\binom{2}{1}}{\binom{4}{2}} = \frac{4}{6} \quad \text{and} \quad P(X=2) = \frac{\binom{2}{2}\binom{2}{0}}{\binom{4}{2}} = \frac{1}{6}.$$

The conditional probability we are looking for is

$$P(X=2 \,|\, X \geq 1) = \frac{P(\text{“}X=2\text{”} \cap \text{“}X \geq 1\text{”})}{P(X \geq 1)} = \frac{P(X=2)}{1-P(X=0)} = \frac{1}{6-1} = 20\%.$$

Observe that this is slightly higher than the unconditional probability $P(X=2) \approx 16.67\%$. That is, by knowing that there is “at least one orange ball,” your estimation of the probability of “two orange balls” should go up from 16.67% to 20%.

**1.3-11. The Birthday Problem.** Consider a classroom containing $r$ students. Assume that each student has a birthday which we can encode as a number from the set $\{1, 2, 3, \ldots, 365\}$ (we ignore leap years), and suppose furthermore that each of these birthdays is equally likely.

(a) Suppose that the $r$ students are ordered (for example, in alphabetical order by last name). If we ask each student for their birthday, what is the size of the sample space? *Answer:*

$$\#S = \underbrace{365}_{\text{1st student's birthday}} \times \underbrace{365}_{\text{2nd student's birthday}} \times \cdots \times \underbrace{365}_{r\text{th student's birthday}} = 365^r.$$

(b) Now consider the event $E =$“no two students have the same birthday.” If $r > 365$ then we are guaranteed that there must be two students with the same birthday, so that $\#E = 0$. Otherwise, if $r \leq 365$ then we have

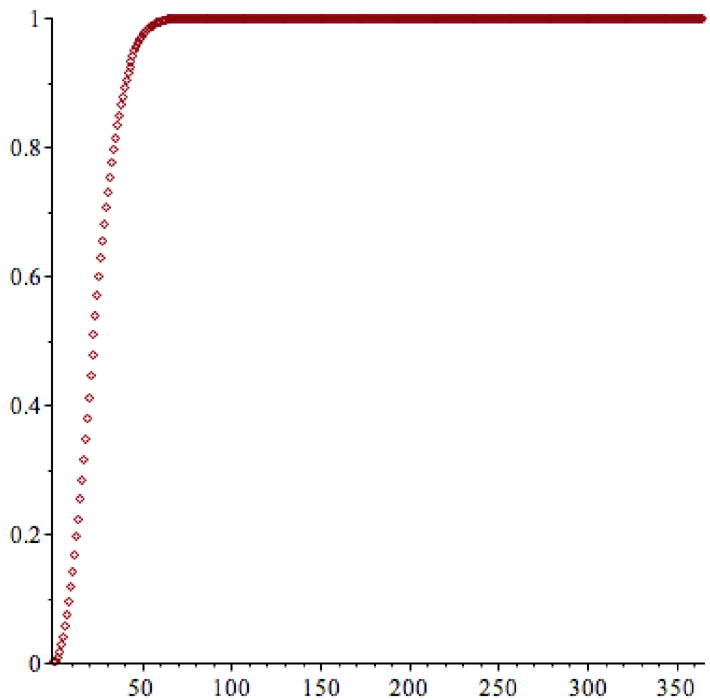
$$\#E = \underbrace{365}_{\text{1st student's birthday}} \times \underbrace{364}_{\text{2nd student's birthday}} \times \cdots \times \underbrace{(365-r+1)}_{r\text{th student's birthday}} = 365!/(365-r)!.$$

(c) Assuming that all outcomes are equally likely, what is the probability that in a class of $r$ students at least two will have the same birthday? *Answer:* If $r \leq 365$ then

$$\begin{aligned} P(\text{at least two share a birthday}) &= 1 - P(\text{no two share a birthday}) \\ &= 1 - P(E) \\ &= 1 - \frac{\#E}{\#S} \\ &= 1 - \frac{365!/(365-r)!}{365^r}. \end{aligned}$$

If $r > 365$ then $P(\text{at least two share a birthday}) = 1 - P(E) = 1 - 0 = 1$.

(d) Here is a plot of the probabilites $1 - P(E)$ for values of $r$ from 1 to 365. Note that the probability rises from 0% when $r = 1$ to 100% when $r = 366$.




At some point the probability must cross 50% and it seems from the diagram that this happens around $r = 25$. To be precise, I used my computer to find the following:

- For $r = 22$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-22)!}{365^{22}} \approx 47.57\%.$$
- For $r = 23$ students, the probability that at least two share a birthday is
$$1 - P(E) = 1 - \frac{365!/(365-23)!}{365^{23}} \approx 50.73\%.$$

Do you find the number 23 surprisingly small? That's why this problem is sometimes also called the **birthday paradox**.

**1.5-2.** Bean seeds come from two suppliers, called $A$ and $B$. Seeds from supplier $A$ have an 85% germination rate and seeds from supplier $B$ have a 75% germination rate. A seed-packing company purchases 40% of its seeds from supplier $A$ and 60% of its seeds from supplier $B$ and mixes them together (uniformly).

(a) You buy a seed from this seed-packing company and plant it. Let $G$ be the event that the seed germinates. Compute $P(G)$. *Answer:* We are given the probabilities

$$\begin{aligned} P(G|A) &= 0.85, \\ P(G|B) &= 0.75, \\ P(A) &= 0.40, \\ P(B) &= 0.60. \end{aligned}$$

In order to compute $P(G)$ we first divide into disjoint pieces using $A$ and $B$:

$$\begin{aligned} G &= (G \cap A) \sqcup (G \cap B) \\ P(G) &= P(G \cap A) + P(G \cap B). \end{aligned}$$

Then we use the definition of conditional probability to obtain

$$\begin{aligned} P(G) &= P(G \cap A) + P(G \cap B) \\ &= P(A)P(G|A) + P(B)P(G|B) \\ &= (0.40)(0.85) + (0.60)(0.75) = 79\%. \end{aligned}$$

(b) Given that the seed germinates, find the probability that the seed was purchased from supplier $A$. *Answer:* We are looking for the probability $P(A|G)$, which we can compute using Bayes' Theorem. In other words, we use the definition of conditional probability together with the result of part (a) to compute

$$\begin{aligned} P(A|G) &= \frac{P(A \cap G)}{P(G)} \\ &= \frac{P(A)P(G|A)}{P(A)P(G|A) + P(B)P(G|B)} \\ &= \frac{(0.40)(0.85)}{(0.40)(0.85) + (0.60)(0.75)} \approx 43.04\%. \end{aligned}$$

**1.5-4.** Drivers are divided into four age ranges:

$$\begin{aligned} R_1 &= \text{“ages 16–25,”} \\ R_2 &= \text{“ages 26–50,”} \\ R_3 &= \text{“ages 51–65,”} \\ R_4 &= \text{“ages 66–90.”} \end{aligned}$$

If a driver is selected at random we are given the probabilities

$$P(R_1) = 0.10, \quad P(R_2) = 0.55, \quad P(R_3) = 0.20 \quad \text{and} \quad P(R_4) = 0.15.$$

[Since these probabilities add to 1, we observe that there are no drivers of age $< 15$ or $> 90$ in this sample.] Now let $A$ be the event that this random driver gets in an accident in a given year. We are given the probabilities

$$P(A|R_1) = 0.05, \quad P(A|R_2) = 0.02, \quad P(A|R_3) = 0.03 \quad \text{and} \quad P(A|R_4) = 0.04.$$

Finally, we can use Bayes' Theorem to compute the conditional probability that a driver who has an accident comes from the $R_1$ age group:

$$\begin{aligned} P(R_1|A) &= \frac{P(R_1)P(A|R_1)}{P(R_1)P(A|R_1) + P(R_2)P(A|R_2) + P(R_3)P(A|R_3) + P(R_4)P(A|R_4)} \\ &= \frac{(0.10)(0.05)}{(0.10)(0.05) + (0.55)(0.02) + (0,20)(0.03) + (0.15)(0.04)} \approx 17.86\%. \end{aligned}$$

Note that this number 17.86% is higher than the proportion $R_1$ drivers in the population (i.e., 10%) because the $R_1$ drivers get in more accidents.

**Additional Problems.**

**1. Pascal's Triangle.** We showed in class that the binomial coefficient $\binom{n}{k}$ for $0 \leq k \leq n$ is given by the formula

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}.$$

When $0 < k < n$, use this formula to prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

*Proof:* By definition, the right hand side is equal to

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]!} + \frac{(n-1)!}{k!\,[(n-1)-k]!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}.\end{aligned}$$

In order to add these fractions we need a common denominator, and the denominator we hope to get is $k!(n-k)!$. So how can we turn $(k-1)!(n-k)!$ and $k!(n-k-1)!$ into $k!(n-k)!$? The trick is to notice that for all positive integers $m$ we have

$$\boxed{m(m-1)! = m!}$$

which, in the cases $m = k$ and $m = n-k$ gives

$$\begin{aligned} k(k-1)! &= k! \\ (n-k)(n-k-1)! &= (n-k)!.\end{aligned}$$

Now we know what to do: We multiplfy the first fraction top and bottom by $k$ and multiply the second fraction top and bottom by $(n-k)$ to obtain

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \cdot \frac{(n-k)}{(n-k)} \\ &= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{[\cancel{k} + (n-\cancel{k})](n-1)!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!},\end{aligned}$$

which equals the left hand side, as desired. ///

**2. Pascal's Tetrahedron.** Let $k_1, k_2, k_3$ be non-negative whole numbers that add to $n$. We saw in class that the *trinomial coefficient* $\binom{n}{k_1,k_2,k_3}$ is given by the formula

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1! \cdot k_2! \cdot k_3!}.$$

In the case that $k_1, k_2, k_3$ are strictly positive, use this formula to prove that

$$\binom{n}{k_1, k_2, k_3} = \binom{n-1}{k_1 - 1, k_2, k_3} + \binom{n-1}{k_1, k_2 - 1, k_3} + \binom{n-1}{k_1, k_2, k_3 - 1}.$$

*Proof:* This one looks harder but I think it's actually easier. By definition, the right hand side is

$$\binom{n-1}{k_1-1,k_2,k_3}+\binom{n-1}{k_1,k_2-1,k_3}+\binom{n-1}{k_1,k_2,k_3-1}$$
$$=\frac{(n-1)!}{(k_1-1)!k_2!k_3!}+\frac{(n-1)!}{k_1!(k_2-1)!k_3!}+\frac{(n-1)!}{k_1!k_2!(k_3-1)!}$$

In order to get a common denominator we use the trick $m(m-1)!=m!$ with $m=k_1$, $m=k_2$ and $m=k_3$ to get

$$\begin{aligned}&\frac{(n-1)!}{(k_1-1)!k_2!k_3!}+\frac{(n-1)!}{k_1!(k_2-1)!k_3!}+\frac{(n-1)!}{k_1!k_2!(k_3-1)!}\\ &=\frac{k_1}{k_1}\cdot\frac{(n-1)!}{(k_1-1)!k_2!k_3!}+\frac{k_2}{k_2}\cdot\frac{(n-1)!}{k_1!(k_2-1)!k_3!}+\frac{k_3}{k_3}\cdot\frac{(n-1)!}{k_1!k_2!(k_3-1)!}\\ &=\frac{k_1(n-1)!}{k_1!k_2!k_3!}+\frac{k_2(n-1)!}{k_1!k_2!k_3!}+\frac{k_3(n-1)!}{k_1!k_2!k_3!}\\ &=\frac{[k_1+k_2+k_3]\,(n-1)!}{k_1!k_2!k_3!}.\end{aligned}$$

Finally, we use the facts $k_1+k_2+k_3=n$ and $n(n-1)!=n!$ to obtain

$$\frac{[k_1+k_2+k_3]\,(n-1)!}{k_1!k_2!k_3!}=\frac{n(n-1)!}{k_1!k_2!k_3!}=\frac{n!}{k_1!k_2!k_3!},$$

which equals the left hand side, as desired. ///

[Remark: When a trinomial power such as $(a+b+c)^n$ is expanded, one can arrange the terms in the shape of a triangle. For example:

$$(a+b+c)^3=\begin{array}{ccccccc} & & & a^3 & & & \\ & & +3a^2b & & +3a^2c & & \\ & +3ab^2 & & +6abc & & +3ac^2 & \\ +b^3 & & +3b^2c & & +3bc^2 & & +c^3\end{array}$$

Thus the trinomial coefficients form a triangle of numbers:

$$\begin{array}{ccccccc} & & & \binom{3}{3,0,0} & & & \\ & & \binom{3}{2,1,0} & & \binom{3}{2,0,1} & & \\ & \binom{3}{1,2,0} & & \binom{3}{1,1,1} & & \binom{3}{1,0,2} & \\ \binom{3}{0,3,0} & & \binom{3}{0,2,1} & & \binom{3}{0,1,2} & & \binom{3}{0,0,3}\end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 6 & & 3 & \\ 1 & & 3 & & 3 & & 1\end{array}$$

One can stack these triangles into the shape of a triangular pyramid in which each number equals the sum of the three numbers directly above. Try it!]