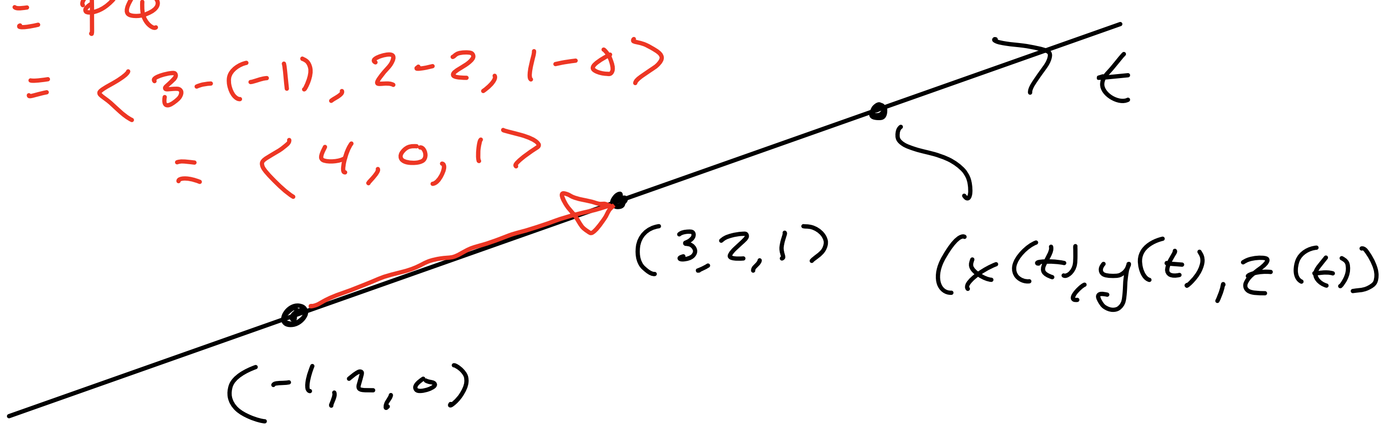
Quiz 2 on Tuesday.
(No class on Monday).

Quiz 2 will cover the material of HW 2. (Material of Thursday's lecture will be on HW 3 & Quiz 3).

Problem 1: Consider line through

$$P = (-1,2,0) \quad \& \quad Q = (3,2,1).$$

$$\vec{v} = \overrightarrow{PQ}$$
$$= \langle 3-(-1),\ 2-2,\ 1-0 \rangle$$
$$= \langle 4,0,1 \rangle$$




$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$
$$= P + t\,\vec{v}$$
$$= \langle -1,2,0 \rangle + t\,\langle 4,0,1 \rangle$$
$$= \langle -1+4t,\ 2+0t,\ 0+1t \rangle$$

$= \langle -1+4t, 2, t \rangle.$

Find two planes whose intersection is this line. Start with

$$\begin{cases} x = -1+4t & \longrightarrow t = (x+1)/4 \\ y = 2 & \longrightarrow t = ? \\ z = t & \longrightarrow t = z \end{cases}$$

Try to eliminate $t$:

Get one plane from

$$(x+1)/4 = t = z.$$

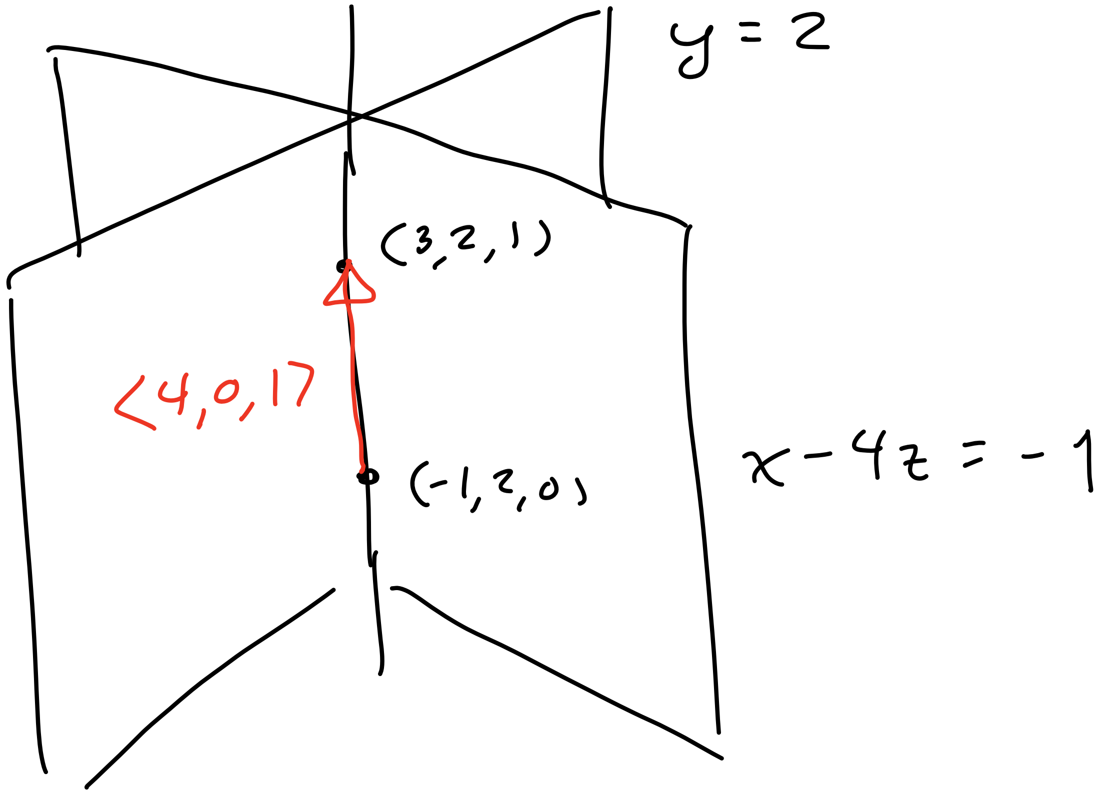
$$x+1 = 4z$$

$$x - 4z = -1$$

Now I need another plane that contains the line.

Idea: We know that $y = 2$, which is a plane.

Take $x - 4z = -1$ & $y = 2$.




We can find infinitely many planes containing this line by taking "linear combinations" of these two planes. For any scalars $k$ & $\ell$ consider the equation

$$
\begin{array}{l}
k\,(x - 4z = -1) \\
+\ \ell\,(y = 2) \\
\hline
k(x-4z) + \ell y = -k + 2\ell .
\end{array}
$$

For any values of $k$ & $l$, this is the equation of a plane that contains our line.

e.g. $k = 1$ & $l = 1$ :

$$(x - 4z) + y = -1 + 2$$

$$x + y - 4z = 1 .$$

Check : $P = (-1, 2, 0)$

$Q = (3, 2, 1)$

$(-1) + (2) - 4(0) = 1$ ✓

$(3) + (2) - 4(1) = 1$ ✓

It works !

Problem 2: Consider two planes

$$\begin{cases} ① & x - y + 2z = 1, \\ ② & 2x + y + 3z = 0. \end{cases}$$

Consider equation ③ = ② − 2 · ①

$$\begin{array}{r} (2\cancel{x} + y + 3z = 0) \\ -2(\cancel{x} - y + 2z = 1) \\ \hline 0 + 3y - z = -2 \end{array} \leftarrow ③$$

Good. We have an equation with no $x$. Now we want an equation with no $y$.

$$\begin{cases} ① & x - y + 2z = 1 \\ ② & 2x + y + 3z = 0 \\ ③ & 3y - z = -2 \end{cases}$$

Any ideas? Combine ① & ②.

Let ④ = ① + ②:

$$
\begin{array}{r}
(x - \cancel{y} + 2z = 1) \\
+ \; (2x + \cancel{y} + 3z = 0) \\
\hline
3x + 0 + 5z = 1 \quad \longleftarrow ④
\end{array}
$$

Now solve for $x$ & $y$ in terms of $z$.

③: $3y - z = -2$

$$3y = -2 + z$$

$$y = -\frac{2}{3} + \frac{1}{3} z.$$

④: $3x + 5z = 1$

$$3x = 1 - 5z$$

$$x = \frac{1}{3} - \frac{5}{3} z.$$

Finally, call $t = z$ a parameter:

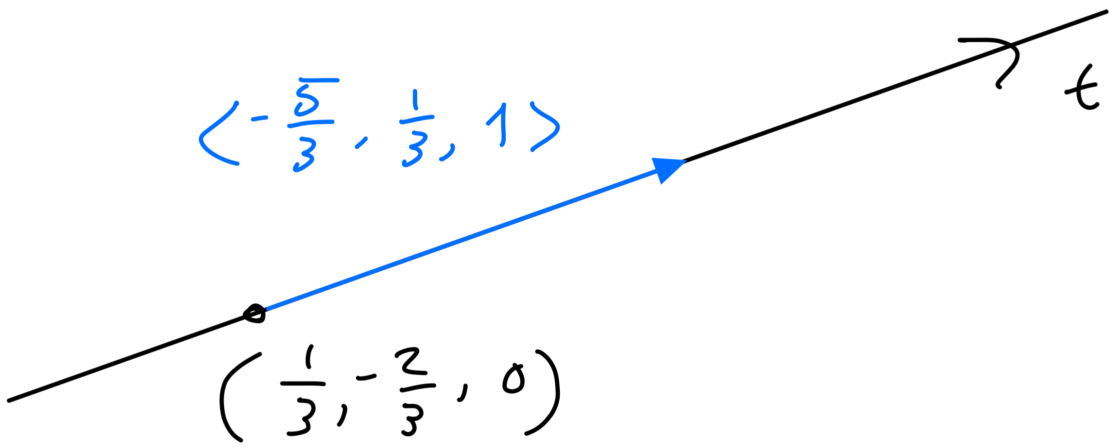
$$
\begin{cases}
x = 1/3 - 5/3\, t \\
y = -2/3 + 1/3\, t \\
z = t.
\end{cases}
$$

Gives a parametrized line

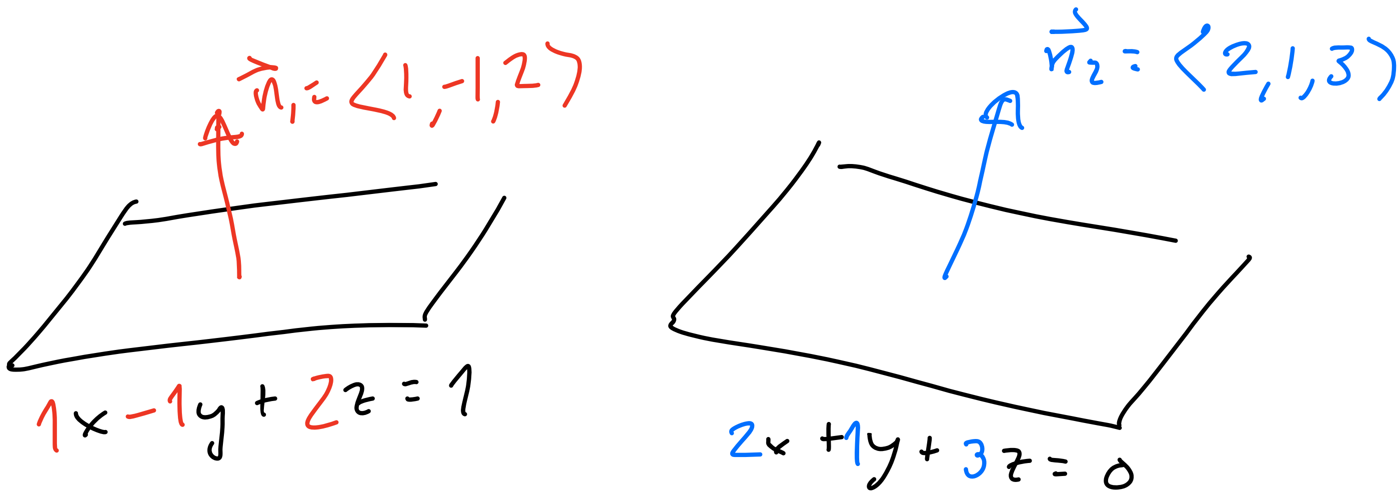
$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$= \langle \tfrac{1}{3} - \tfrac{5}{3}t, -\tfrac{2}{3} + \tfrac{1}{3}t, 0 + 1t \rangle$$

$$= \langle \tfrac{1}{3}, -\tfrac{2}{3}, 0 \rangle + t \langle -\tfrac{5}{3}, \tfrac{1}{3}, 1 \rangle$$




(b) Consider the normal vectors:




Take cross product:

$$\vec{n}_1 \times \vec{n}_2 = \det \begin{pmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \det \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} i - \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} j$$

$$+ \det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} k$$

$$= (-3-2)\, i - (3-4)\, j$$

$$+ (1-(-2))\, k$$

$$= -5i + 1j + 3k$$

$$= \langle -5, 1, 3 \rangle .$$

OBSERVE:

$$\langle -5, 1, 3 \rangle = 3 \left\langle -\frac{5}{3}, \frac{1}{3}, 1 \right\rangle .$$

So this vector is parallel to the line of intersection.

In general: Given planes

$$\vec{n}_1 \cdot \langle x, y, z \rangle = c_1$$
$$\vec{n}_2 \cdot \langle x, y, z \rangle = c_2$$

who cares

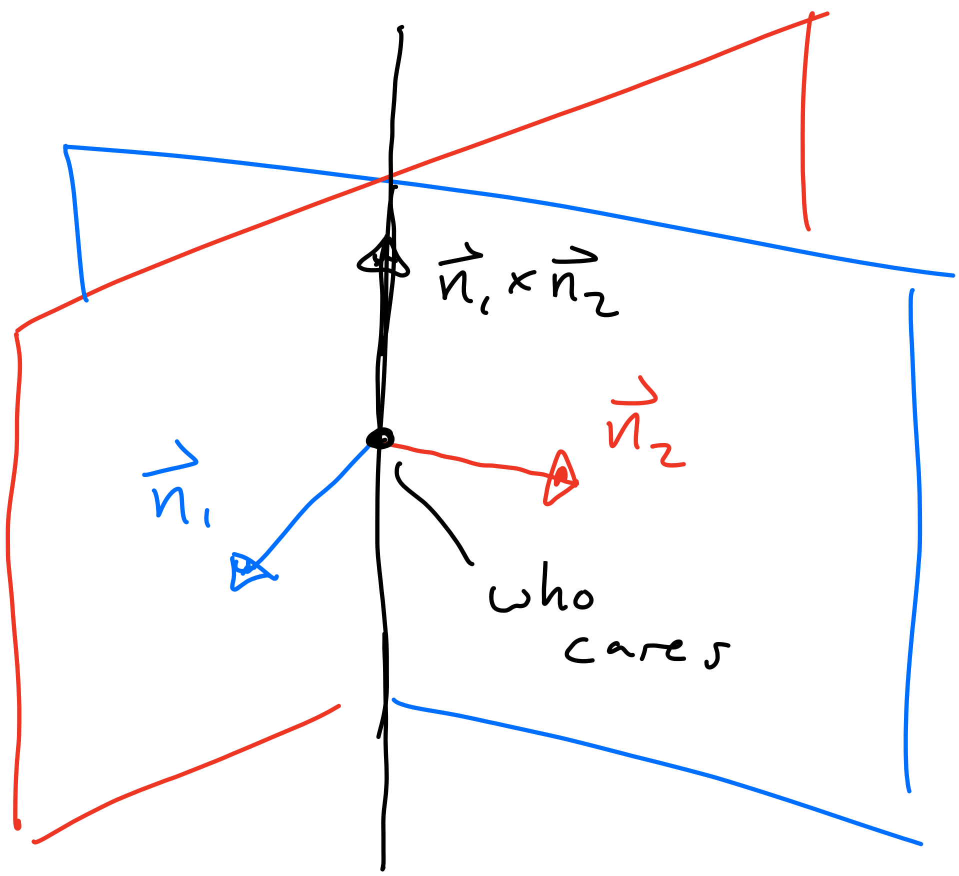
the line of intersection has form

who cares

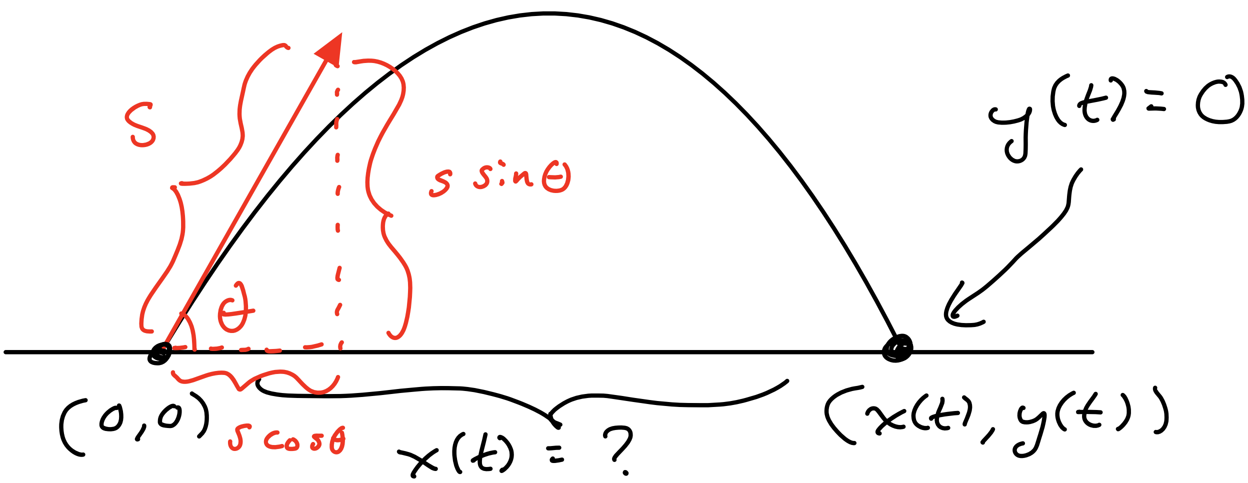
$$\vec{r}(t) = P + t\,(\vec{n}_1 \times \vec{n}_2)$$

For some point $P$.

Picture:



# Problem 3: Projectile Motion.




Initial position = $(0,0)$

Initial velocity = $\langle s\cos\theta, s\sin\theta \rangle$

speed $s$

angle $\theta$.

$$\vec{a}(t) = \langle 0, -g \rangle \quad \text{constant}$$

$$\vec{v}(t) = \langle c_1, -gt + c_2 \rangle$$

$$\vec{v}(0) = \langle c_1, c_2 \rangle = \langle s\cos\theta, s\sin\theta \rangle$$

$$\vec{v}(t) = \langle \underbrace{s\cos\theta}_{\text{const.}}, -gt + \underbrace{s\sin\theta}_{\text{const.}} \rangle$$

$\vec{r}(t)$

$$= \left\langle (s\cos\theta)t + c_3,\ -\frac{1}{2}gt^2 + (s\sin\theta)t + c_4 \right\rangle$$

$$\vec{r}(0) = \langle c_3, c_4 \rangle = \langle 0, 0 \rangle.$$

Conclusion:

$$\vec{r}(t) = \Big\langle \underbrace{(s\cos\theta)t}_{x(t)},\ \underbrace{-\frac{1}{2}gt^2 + (s\sin\theta)t}_{y(t)} \Big\rangle$$

Next: Set $y(t) = 0$

$$-\frac{1}{2}gt^2 + (s\sin\theta)t = 0$$

$$t\left(-\frac{1}{2}gt + s\sin\theta\right) = 0$$

$$t = 0 \quad \text{or} \quad -\frac{1}{2}gt + s\sin\theta = 0$$

$$\frac{1}{2}gt = s\sin\theta$$

$$t = \frac{2s\sin\theta}{g}$$

DONE.

How far did it travel in horizontal direction?

$$x(t) = (s\cos\theta)t$$

$$= (s\cos\theta)\cdot\frac{2s\sin\theta}{g}$$

$$= \frac{2s^2}{g}\cos\theta\sin\theta$$

For which angle $\theta$ is this maximized?

Think of it as a function of $\theta$:

$$f(\theta) = \frac{2s^2}{g}\cos\theta\sin\theta$$

$$= \frac{s^2}{g}\sin(2\theta).$$

$$\frac{df}{d\theta} = \frac{\partial f}{\partial\theta} = \frac{s^2}{g}\cos(2\theta)\cdot 2$$

$$\frac{df}{d\theta} = 0$$

$$\cancel{\frac{2s^2}{g}} \cos(2\theta) = 0$$

$$\cos(2\theta) = 0$$

$$2\theta = 90^\circ \text{ or } \cancel{-90^\circ}$$

$$\boxed{\theta = 45^\circ}$$

Distance maximized when you aim the cannon at $45^\circ$.

INDEPENDENT of $s$ & $g$.

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Problem 4. Vector Identities.

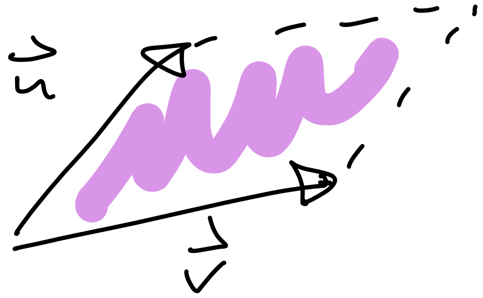
(a) $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{v} \times \vec{v} = \det \begin{pmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

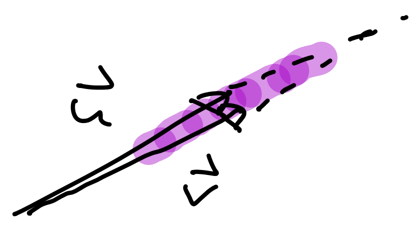
$$= \langle v_2v_3 - v_2v_3,\ v_3v_1 - v_1v_3,\ v_1v_2 - v_2v_1 \rangle$$

$= \langle 0, 0, 0 \rangle.$

Alternatively: The area of parallelogram generated by $\vec{v}$ & $\vec{v}$ is zero.




area $= \|\vec{u} \times \vec{v}\|$




$\vec{u}$ close to $\vec{v}$, area close to zero.

So $\|\vec{v} \times \vec{v}\| = 0$

$$\implies \vec{v} \times \vec{v} = \vec{0}$$

Alternatively:

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta.$$

$$\|\vec{v} \times \vec{v}\| = \|\vec{v}\| \|\vec{v}\| \sin(0) = 0.$$

(b) Given vector $\vec{r}$, find a vetor $\alpha \vec{r}$ with length 1.

$$\|\alpha \vec{r}\| = |\alpha|\,\|\vec{r}\| = 1$$

$$|\alpha| = \frac{1}{\|\vec{r}\|}$$

$$\alpha = \frac{1}{\|\vec{r}\|}$$

So the vector $\underbrace{\frac{1}{\|\vec{r}\|}}_{\text{scalar}}\underbrace{\vec{r}}_{\text{vector.}}$ has length 1.

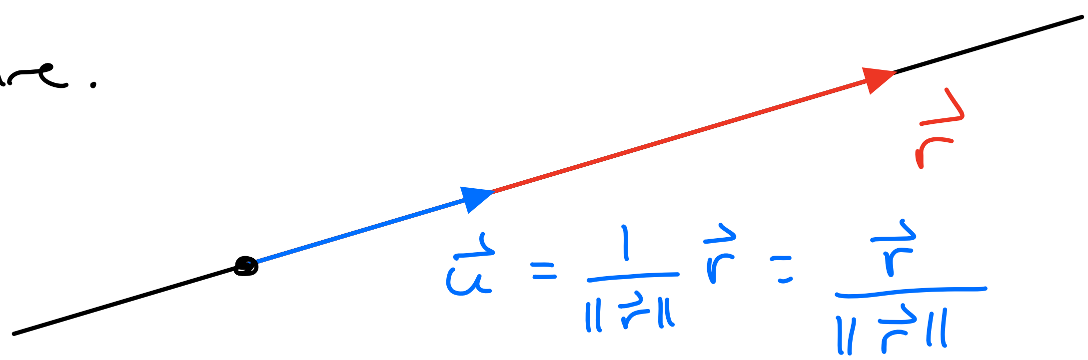
Alternatively:

$$\left\|\frac{1}{\|\vec{r}\|}\vec{r}\right\|^2 = \left(\frac{1}{\|\vec{r}\|}\vec{r}\right)\cdot\left(\frac{1}{\|\vec{r}\|}\vec{r}\right)$$

$$= \left(\frac{1}{\|\vec{r}\|}\right)^2 \vec{r}\cdot\vec{r}$$

$$= \left(\frac{1}{\cancel{\|\vec{r}\|^2}}\right)\cancel{\|\vec{r}\|^2} = 1.$$

Picture.



$\vec{u}$ is the unit vector in the direction of $\vec{r}$.

(c) Suppose $\|\vec{r}(t)\| =$ constant.

i.e. the particle travels on the surface of a sphere (in some # of dimensions). Then

$$\|\vec{r}(t)\|^2 = \text{constant}$$

$$\vec{r}(t) \cdot \vec{r}(t) = \text{constant}$$

$$\left[\vec{r}(t) \cdot \vec{r}(t)\right]' = 0$$

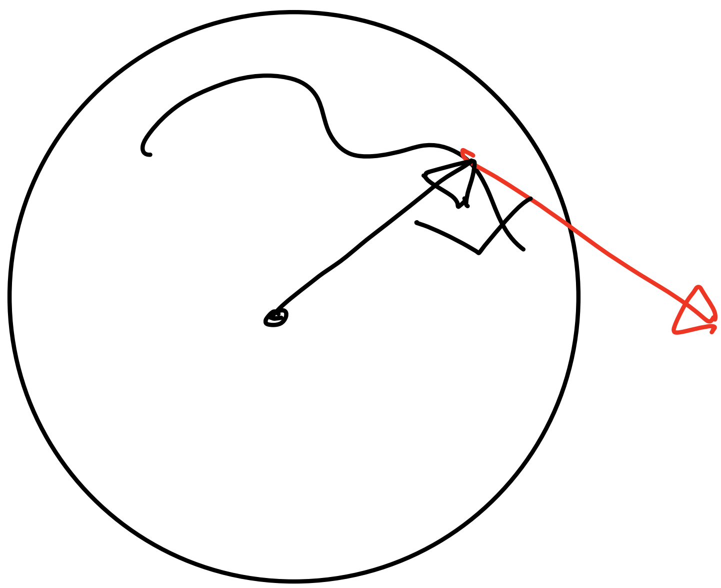
$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

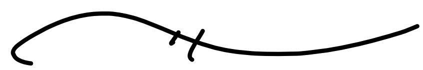
$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

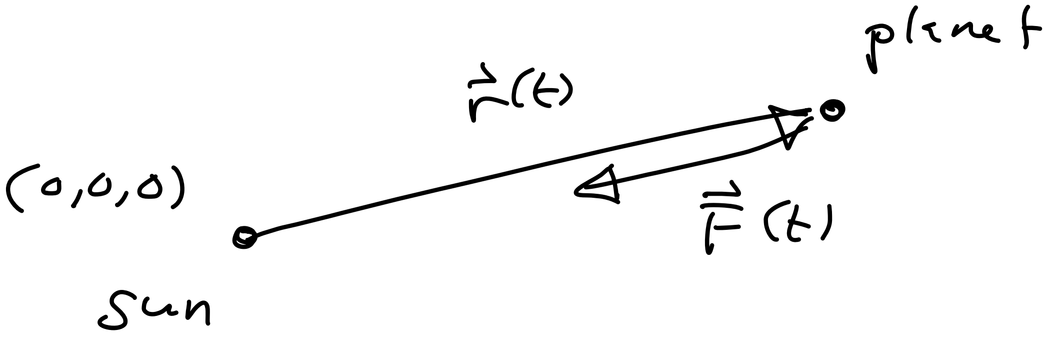
Position is $\perp$ to velocity at all times $t$.

Meaning: velocity is tangent to the sphere





Problem 5: Universal Gravitation.




$$\vec{F}(t) = -\alpha\, \vec{r}(t) \quad \text{for some } \alpha > 0.$$

$$\|\vec{F}(t)\| = GMm / \|\vec{r}(t)\|^2$$

Put together

$$\|-\alpha \vec{r}(t)\| = \alpha \|\vec{r}(t)\|.$$

$$\frac{GMm}{\|\vec{r}(t)\|^2} = \alpha \|\vec{r}(t)\|$$

$$\alpha = \frac{GMm}{\|\vec{r}(t)\|^3}$$

$$\vec{F}(t) = -\alpha \vec{r}(t)$$

$$\vec{F}(t) = -\frac{GMm}{\|\vec{r}(t)\|^3} \vec{r}(t)$$

Newton's 2nd Law

$$\vec{F}(t) = m \vec{r}''(t).$$

$$\boxed{\vec{r}''(t) = -\frac{GM}{\|\vec{r}(t)\|} \vec{r}(t)}$$

System of 2nd order diff eq's

describing the orbit

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

Goal: SOLVE for $\vec{r}(t)$.

Much harder than projectile motion near the surface of Earth.

We'll just do step 1:

Conservation of angular momentum.

$$\vec{L}(t) = \vec{r}(t) \times \vec{r}\,'(t)$$

position $\times$ velocity.

$$\vec{L}\,'(t) = \left( \vec{r}(t) \times \vec{r}\,'(t) \right)'$$

$$= \cancel{\vec{r}\,'(t) \times \vec{r}\,'(t)} + \vec{r}(t) \times \vec{r}\,''(t).$$

$$= \langle 0,0,0 \rangle + \vec{r}(t) \times \left( \frac{-GM}{\|\vec{r}(t)\|^3} \vec{r}(t) \right)$$

$$= \langle 0,0,0 \rangle - \frac{GM}{\|\vec{r}(t)\|^3} \cancel{\vec{r}(t) \times \vec{r}(t)}$$

$$= \langle 0,0,0 \rangle - \frac{GM}{\|\vec{r}(t)\|^3} \langle 0,0,0 \rangle$$

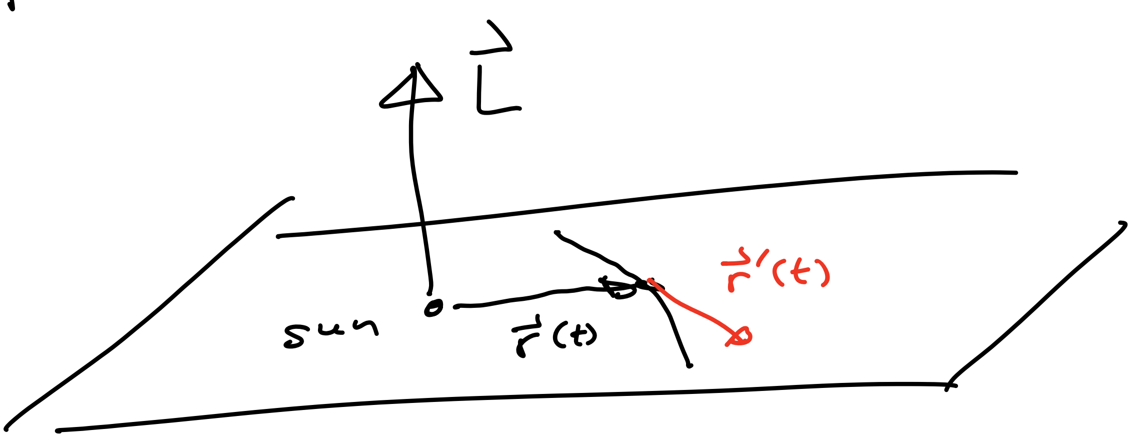
$$= \langle 0,0,0 \rangle .$$

So $\vec{L}'(t) = \langle 0,0,0 \rangle$

hence $\vec{L}(t)$ is a constant vector.

One consequence:

The planet orbits in a fixed plane $\perp$ to $\vec{L}$.




Planet stays in the plane