HW 2 due Friday before class.

Motion in Space.

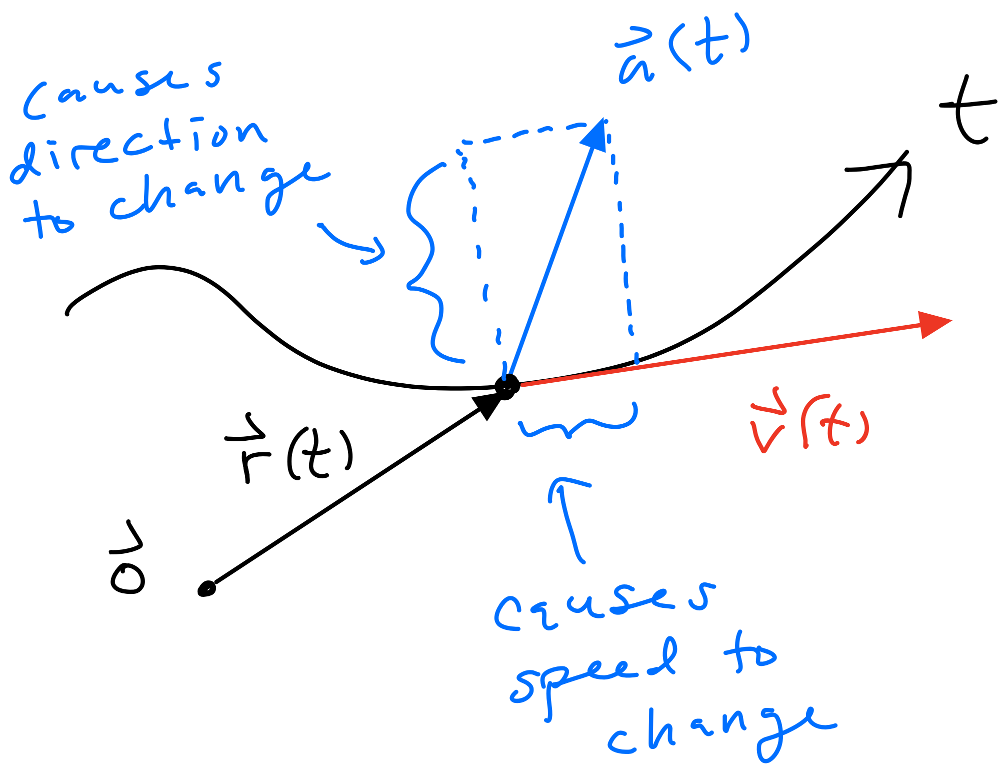
Given parametrized curve in $\mathbb{R}^3$:

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\vec{a}(t) = \vec{v}\,'(t)$$
$$= \vec{r}\,''(t) = \langle x''(t), y''(t), z''(t) \rangle.$$

Picture:



Newton's Second Law:

force = mass · acceleration.

If a force $\vec{F}(t)$ acts on a particle will mass m & position $\vec{r}(t)$ then we have

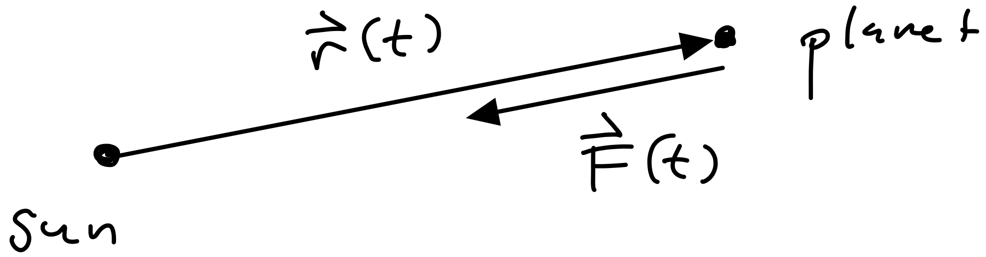
$$\vec{F}(t) = m\,\vec{r}\,''(t)$$

Example: Gravity.

The sun is at origin $(0,0,0)$ in $\mathbb{R}^3$.
A planet has position $\vec{r}(t)$.
Let $\vec{F}(t)$ be the gravitational force felt by the planet. Then:

- $\vec{F}(t)$ points directly toward the sun.
- $\|\vec{F}(t)\| = GMm / \|\vec{r}(t)\|^2$

where $G =$ gravitational constant
$M =$ mass of sun
$m =$ mass of planet.



$\vec{F}(t) = ?$

Know : $\vec{F}(t) = -c(t)\,\vec{r}(t)$

for some scalar $c(t)$.

Use the fact that

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|.$$

[Proof : $\|c\vec{v}\|^2 = (c\vec{v}) \cdot (c\vec{v})$

$$= c^2\, \vec{v} \cdot \vec{v}$$

$$= c^2 \|\vec{v}\|^2$$

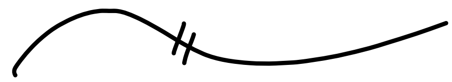
$$\|c\vec{v}\| = \sqrt{c^2 \|\vec{v}\|^2}$$

$$= |c|\,\|\vec{v}\|. \quad]$$

Know $\|\vec{F}(t)\| = GMm / \|\vec{r}(t)\|^2$.

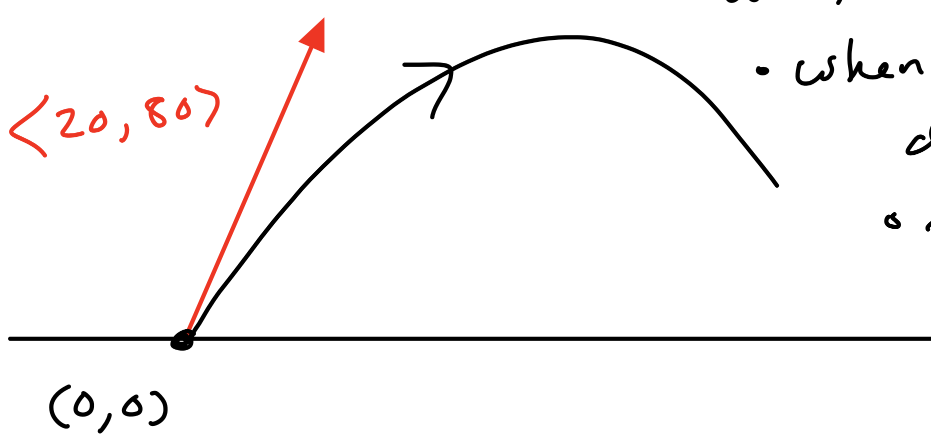
And $\|\vec{F}(t)\| = |c(t)|\,\|\vec{r}(t)\|$

So $c(t) = ?$ HW 2 Problem 5.



Easier: Projectile Motion near surface of the Earth.

Projectile will travel in a 2D plane, so we'll just describe in the x,y-plane.



- what shape?
- when/where does it land?
- how high?

Galileo: $\vec{r}''(t)$ is <u>constant</u>.

$$\vec{r}\,''(t) = \langle 0, -32 \text{ feet/sec}^2 \rangle$$

[This is a "textbook problem" so the numbers will be nice.]

Integrate to get velocity.

$$\vec{v}(t) = \vec{r}\,'(t) = \int \vec{r}\,''(t)\,dt.$$

$$= \left\langle \int 0\,dt, \int -32\,dt \right\rangle$$

$$= \langle c_1, -32t + c_2 \rangle$$

To find constants, sub $t = 0$:

$$\vec{v}(0) = \langle c_1, -32(0) + c_2 \rangle = \langle c_1, c_2 \rangle$$

GIVEN $\vec{v}(0) = \left\langle 20 \frac{\text{feet}}{\text{sec}}, 80 \frac{\text{feet}}{\text{sec}} \right\rangle.$

So $c_1, c_2 = 20, 80$.

$$\vec{v}(t) = \langle 20, 80 - 32t \rangle.$$

Integrate again to get position:

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \vec{r}\,'(t)\,dt$$

$$= \left\langle \int 20\,dt, \int (80-32t)\,dt \right\rangle$$

$$= \left\langle 20t + c_3,\ 80t - 16t^2 + c_4 \right\rangle$$

To find c_3, c_4, sub $t=0$.

$$\vec{r}(0) = \langle c_3, c_4 \rangle$$

GIVEN $\vec{r}(0) = \langle 0, 0 \rangle$.

So $c_3 = 0$ & $c_4 = 0$ hence

$$\boxed{\vec{r}(t) = \langle 20t,\ 80t - 16t^2 \rangle}$$

Shape? Eliminate t:

$$x = 20t \quad \rightsquigarrow \quad t = x/20.$$

$$y = 80t - 16t^2$$

$$y = 80\left(\frac{x}{20}\right) - 16\left(\frac{x}{20}\right)^2$$

$$y = 4x - \frac{1}{25}x^2$$

It's a parabola!

Where/When does it land?

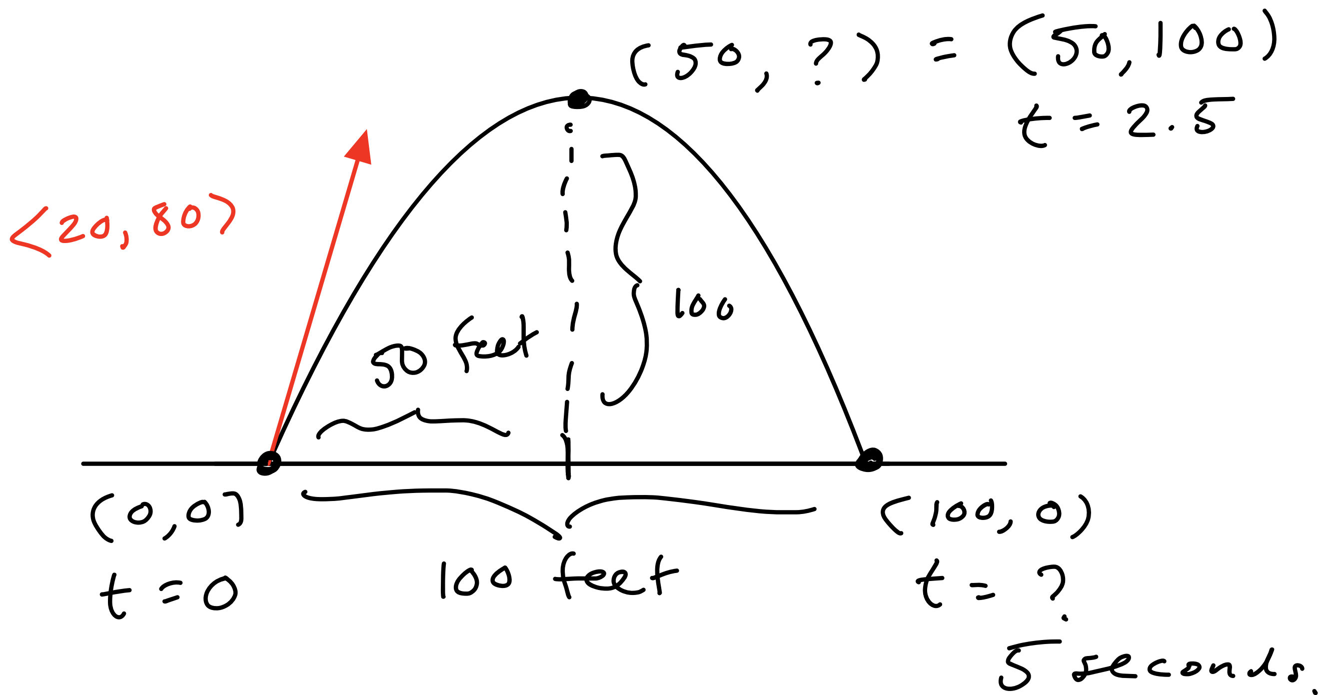
$y = 0$.

$4x - \frac{1}{25}x^2 = 0$

$100x - x^2 = 0$

$x(100 - x) = 0$

$\rightsquigarrow$ $x = 0$ or $x = 100$.



when?

$y(t) = 0$

$80t - 16t^2 = 0$

$t(80 - 16t) = 0$

$t = 0$ or $80 - 16t = 0$

$t = 80/16 = 5$ seconds.

How High ?

Two ways :

- Use $y = 4x - \frac{1}{25}x^2$

Parabolas are symmetric.

slope at top = 0

$dy/dx = 0$

$4 - \frac{2}{25}x = 0$

$x = \frac{4 \cdot 25}{2} = 50$ ✓

So $y = 4(50) - \frac{1}{25}(50)^2$

$= 200 - 2 \cdot 50 = 100$ feet.

- Use $\vec{r}(t) = \langle 20t, 80t - 16t^2 \rangle$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{80 - 32t}{20} = 0.$ Horizontal Tangent.

$80 - 32t = 0$

$t = 80/32 = 2.5$ seconds.

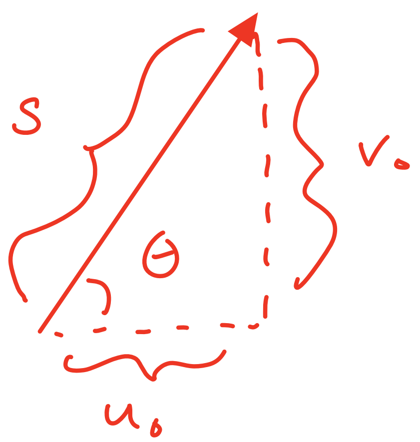
More generally:

$$\vec{r}(0) = \langle x_0, y_0 \rangle$$
$$\vec{v}(0) = \langle u_0, v_0 \rangle$$
$$\vec{a}(t) = \langle 0, -g \rangle$$

Then integrating twice gives

$$\vec{r}(t) = \langle x_0 + u_0 t, y_0 + v_0 t - \tfrac{1}{2} g t^2 \rangle$$

More common to describe $\vec{v}(0)$ in terms of speed and angle.



$u_0 = s \cos\theta$

$v_0 = s \sin\theta$.

$$\vec{r}(t) = \langle x_0 + s\cos\theta \cdot t, y_0 + s\sin\theta \cdot t - \tfrac{1}{2} g t^2 \rangle$$

See HW 2 Problem 3.

Maximize the horizontal distance traveled.

Differentiation Rules:

Consider some vector-valued functions:

$$\vec{u} : \mathbb{R} \to \mathbb{R}^n$$

$$\vec{v} : \mathbb{R} \to \mathbb{R}^n$$

$$\vec{u}(t) = \langle u_1(t), u_2(t), \ldots, u_n(t) \rangle$$

$$\vec{v}(t) = \langle v_1(t), v_2(t), \ldots, v_n(t) \rangle$$

Also consider scalar $c \in \mathbb{R}$ and a regular function

$$f : \mathbb{R} \to \mathbb{R}.$$

Then we have the following rules:

$$[c\,\vec{u}(t)]' = c\,\vec{u}\,'(t)$$

$$[\vec{u}(t) \pm \vec{v}(t)]' = \vec{u}\,'(t) \pm \vec{v}\,'(t)$$

$$[\overbrace{f(t)\,\vec{u}(t)}^{\text{vector}}]' = \underbrace{f'(t)\,\vec{u}(t)}_{\text{vector}} + \underbrace{f(t)\,\vec{u}\,'(t)}_{\text{vector}}.$$

scalar that changes

vector that changes

$$\left[\underbrace{\vec{u}(t)\cdot\vec{v}(t)}_{\text{scalar that changes}}\right]' = \underbrace{\vec{u}\,'(t)\cdot\vec{v}(t)}_{\text{scalar that changes}} + \underbrace{\vec{u}(t)\cdot\vec{v}\,'(t)}_{\text{scalar that changes.}}.$$

$$\left[\underbrace{\vec{u}(f(t))}_{\text{vector}}\right]' = \underbrace{\vec{u}\,'(f(t))}_{\text{vector}}\cdot\underbrace{f'(t)}_{\text{scalar}}$$

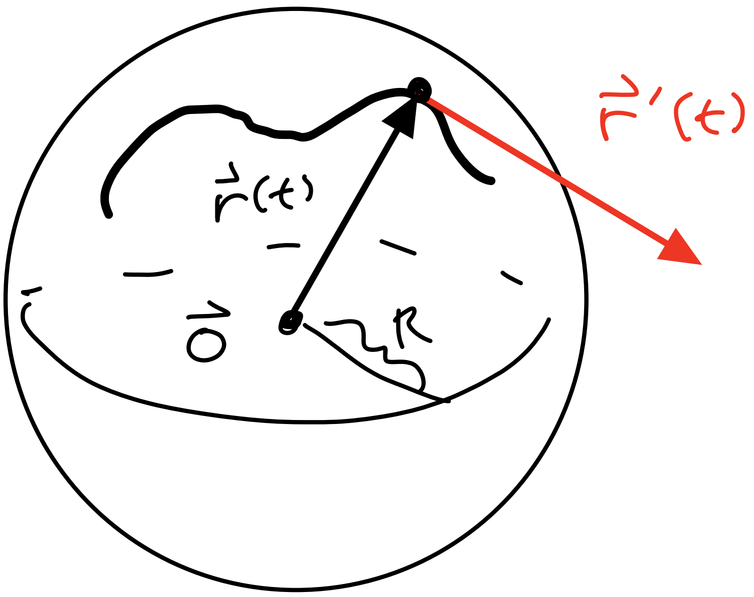
Just like Calc I

If we are working in $\mathbb{R}^3$ then we also have a "product rule" for cross products:

$$\left[\underbrace{\vec{u}(t)\times\vec{v}(t)}_{\text{vector}}\right]' = \underbrace{u'(t)\times\vec{v}(t)}_{\text{vector}} + \underbrace{\vec{u}(t)\times\vec{v}\,'(t)}_{\text{vector}}$$

Good news: Easy to memorize.

Application: Suppose particle travels on surface of a sphere of radius R.



I claim that $\vec{r}(t) \perp \vec{r}'(t)$ for all times t. (The velocity is always tangent to sphere.)

<u>Proof</u>: GIVEN

$$\|\vec{r}(t)\| = R \quad \text{for all } t.$$

$$\|\vec{r}(t)\|^2 = R^2$$

$$\underbrace{\vec{r}(t) \cdot \vec{r}(t)}_{\text{scalar}} = \underbrace{R^2}_{\text{scalar}}$$

Differentiate both sides with resp. to t.

$$[\vec{r}(t) \cdot \vec{r}(t)]' = 0$$

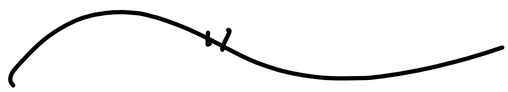
$$\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \quad \checkmark$$

JUST ALGEBRA !

[Recall : $2(\vec{u} \cdot \vec{v}) = (2\vec{u}) \cdot \vec{v} = \vec{u} \cdot (2\vec{v})$]



Preview of next Topic:

How should we think of a function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

For any vector $\vec{v}$ in $\mathbb{R}^2$

we get a scalar $f(\vec{v}) \in \mathbb{R}$.

OR: For any point $P = (x_1, x_2, \ldots, x_n)$

we get a scalar

$$f(x_1, x_2, \ldots, x_n) \in \mathbb{R}.$$

This could represent

- temperature at a point
- pressure
- density
- chemical concentration

$\vdots$

etc.

We are attaching a number to each point in space.

Called a SCALAR FIELD.

How can we visualize this?

Example: The temperature at the point (x, y) in $\mathbb{R}^2$ is

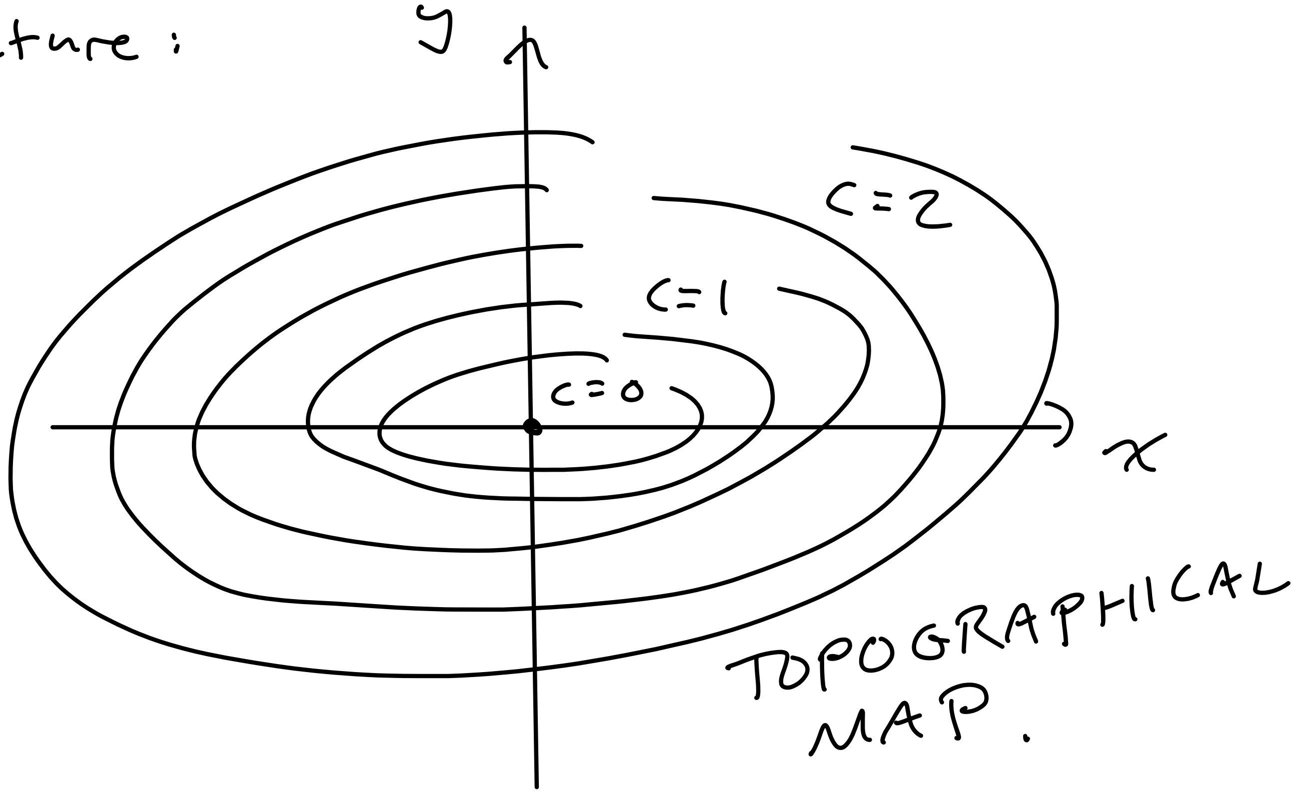
$$f(x,y) = \left(\frac{x}{2}\right)^2 + y^2.$$

For each fixed temperature c, the set of points with this temperature is an ellipse

$$f(x,y) = c$$

$$\left(\frac{x}{2}\right)^2 + y^2 = c.$$

Picture:



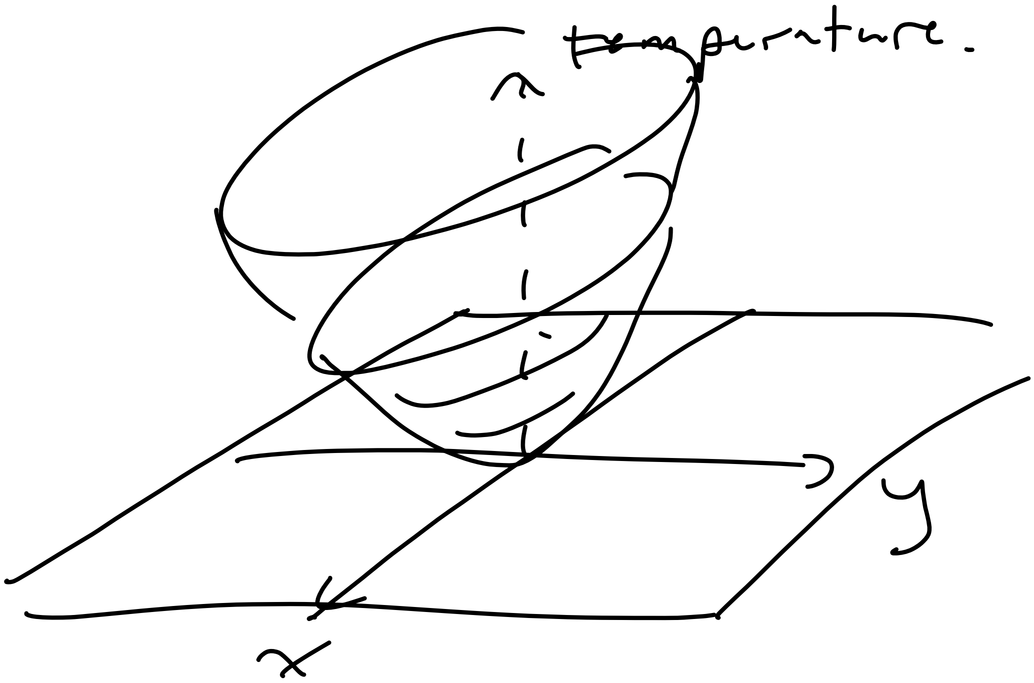
These ellipses called "isotherms" or "curves of constant temperature".

Also call them the "level curves" of the function $f(x,y)$.

Better: Think of temperature as a "third variable".

Then view $f(x,y)$ as a "2D

surface in $\mathbb{R}^3$ "



The surface is a parabolic bowl.
This surface is just the "graph"
of the function $F(x,y)$.