Quiz 1 Discussion:

Problem 1:

$$f(t) = (x(t), y(t)) = (1+3t^2, 4t^2)$$

$$f'(t) = (0 + 3\cdot 2t, 4\cdot 2t)$$

$$= (6t, 8t)$$

$$\|f'(t)\| = \sqrt{(6t)^2 + (8t)^2}$$

$$= \sqrt{36t^2 + 64t^2}$$

$$= \sqrt{100t^2}$$

$$= 10t$$

Arc length $t = 0 .. 1$

$$= \int_0^1 10t \, dt$$

$$= 10\left[\frac{t^2}{2}\right]_0^1 = 5.$$

What does the curve look like?

Eliminate t:

$x = 1 + 3t^2 \longrightarrow t^2 = (x-1)/3$

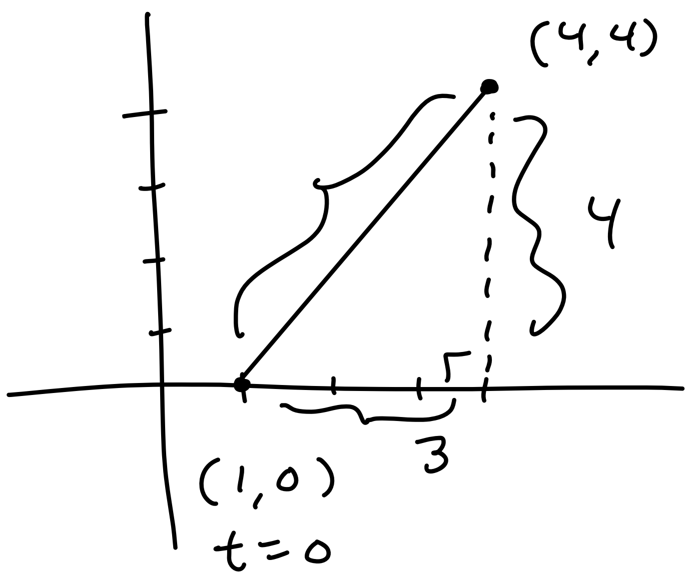
$y = 4t^2 \longrightarrow t^2 = y/4$

$$\frac{(x-1)}{3} = \frac{y}{4}$$

$$4(x-1) = 3y.$$

$$4x - 3y = 4 \quad \text{Line!}$$

Picture:



$\text{Arc Length}^2 = 3^2 + 4^2 = 25$

$\text{Arc Length} = 5$ ✓

Moving on with Chapters 2 & 3.

Sketch:

Chap 2 & 3: Vectors, Vector-valued functions (i.e., parametrized paths).
Motion in space & integration of vector-valued functions.
Lines & Planes.

Chapter 4: Differentiation in any # of dimensions...
In particular, GRADIENTS.

Chapter 5: Integration in any # of dimensions.
e.g. surface area, volume, physics (total work/energy, ...)

Chapter 6: Putting it all together.
Div, Grad, Curl stuff.

Function $f : \mathbb{R} \to \mathbb{R}^2$
or $\mathbb{R} \to \mathbb{R}^3$

Think of as a parametrized curve in $\mathbb{R}^2$ or $\mathbb{R}^3$. New notation:

$$\vec{f} : \mathbb{R} \to \mathbb{R}^3$$
$$\vec{r} : \mathbb{R} \to \mathbb{R}^3$$

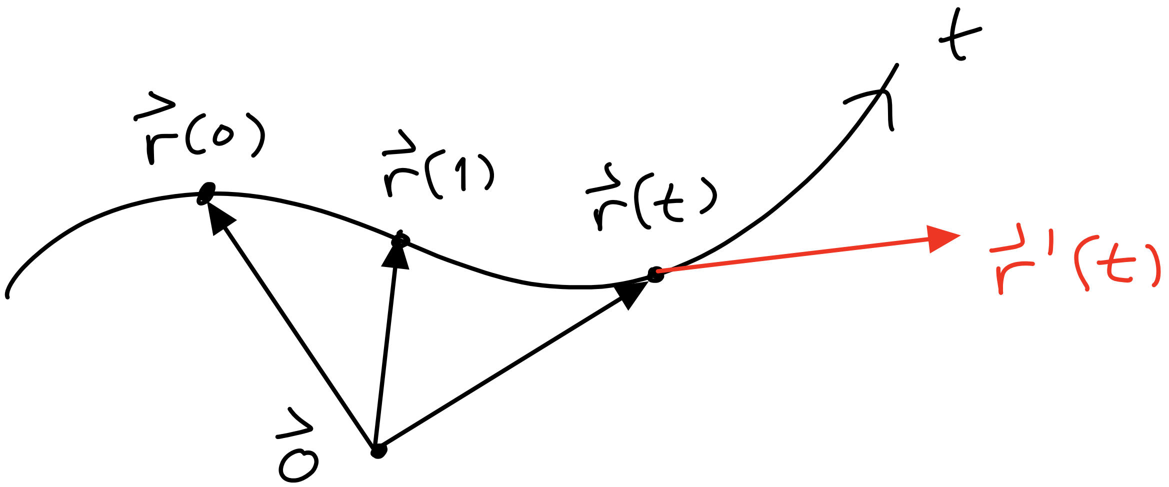
the output is a vector.

Notation

$$\vec{r}(t) = (x(t), y(t), z(t))$$

is common in physics. [I think $\vec{r}$ stands for "radius".]

Picture:



Parametrized line in any # of dimensions:

$$\vec{r}(t) = \vec{x}_0 + t\,\vec{v}$$

initial position (at time 0)

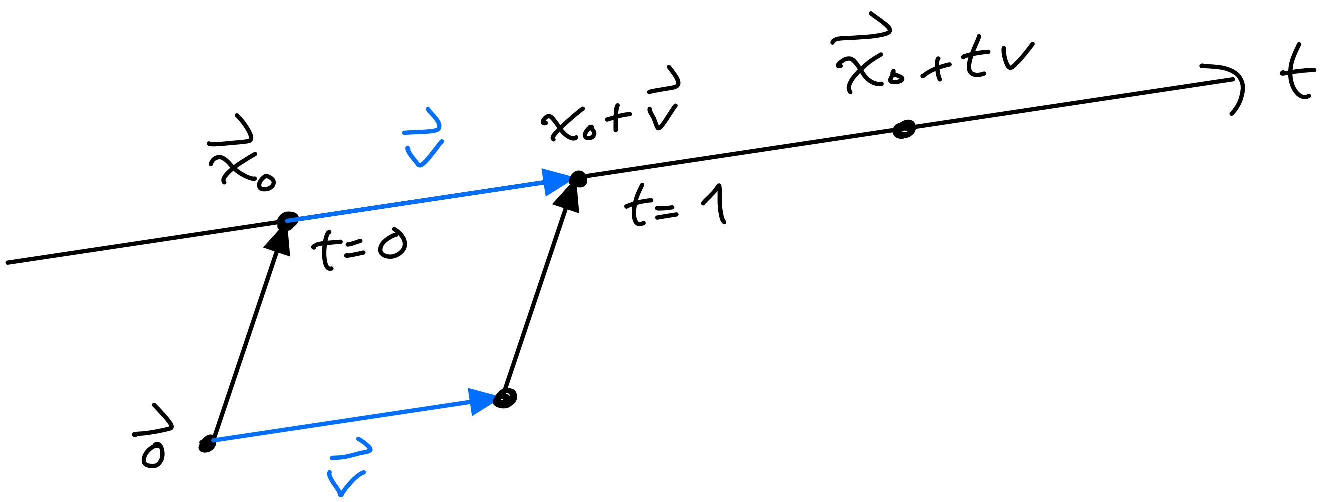
velocity.

e.g. in 3D.

$$\vec{x}_0 = (x_0, y_0, z_0)$$

$$\vec{v} = \langle a, b, c \rangle$$

$$\vec{r}(t) = \langle x_0 + ta,\ y_0 + tb,\ z_0 + tc \rangle$$



We know the equation of a line in $\mathbb{R}^2$ & a plane in $\mathbb{R}^3$:

$$a(x-x_0) + b(y-y_0) = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

What is the equation of a <u>line in $\mathbb{R}^3$</u>?

TRICK QUESTION!

A line in $\mathbb{R}^3$ cannot be described with only one equation. We need at least 2 equations.

e.g. Consider a parametrized line

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}.$$

Try to eliminate t:

$$t = \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

This gives us 3 different equations involving x, y, z (but not t):

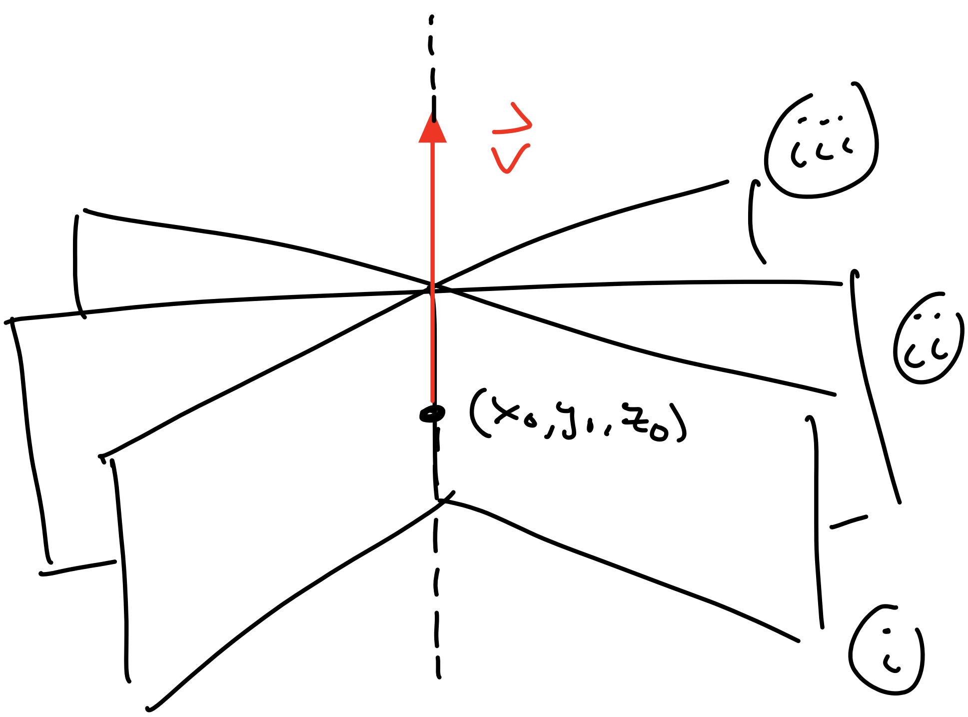
(i) $(x-x_0)/a = (y-y_0)/b$

(ii) $(x-x_0)/a = (z-z_0)/c$

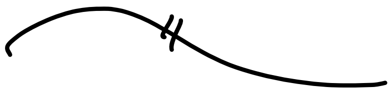
(iii) $(y-y_0)/b = (z-z_0)/c$.

Each of these represents a <u>plane</u>

Any two of these planes intersect at the original line:

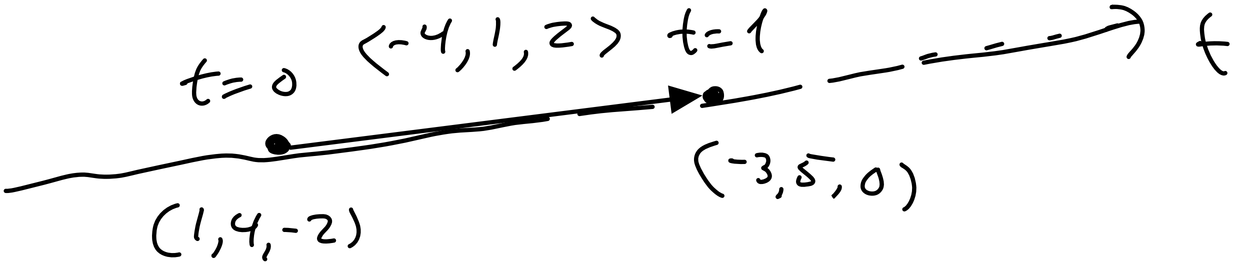


i, ii, iii are called the "symmetric equations" of the line. But there are ∞ many pairs of equations that describe this line.



Example: Find a parametrization & symmetric equations for the line in $\mathbb{R}^3$ through points

$$P = (1, 4, -2) \quad \& \quad Q = (-3, 5, 0).$$



Initial point $\vec{x}_0 = (1, 4, -2)$

velocity $\vec{v} = \langle -4, 1, 2 \rangle$

Parametrization:

$$\vec{r}(t) = (1-4t, 4+t, -2+2t).$$

$$\text{OR} \quad \begin{cases} x = 1-4t \\ y = 4+t \\ z = -2+2t \end{cases}$$

Eliminate t to obtain the symmetric equations:

$$t = \frac{x-1}{-4} = \frac{y-4}{1} = \frac{z+2}{2}$$

So our line is at the intersection of the following 3 planes:

(i) $(x-1)/(-4) = y-4$

$$(x-1) = -4y + 16$$

$$x + 4y = 17$$

(ii) $(x-1)/(-4) = (z+2)/2$

$$2(x-1) = (-4)(z+2)$$

$$2x - 2 = -4z - 8$$

$$2x + 4z = -6$$

$$x + 2z = -3$$

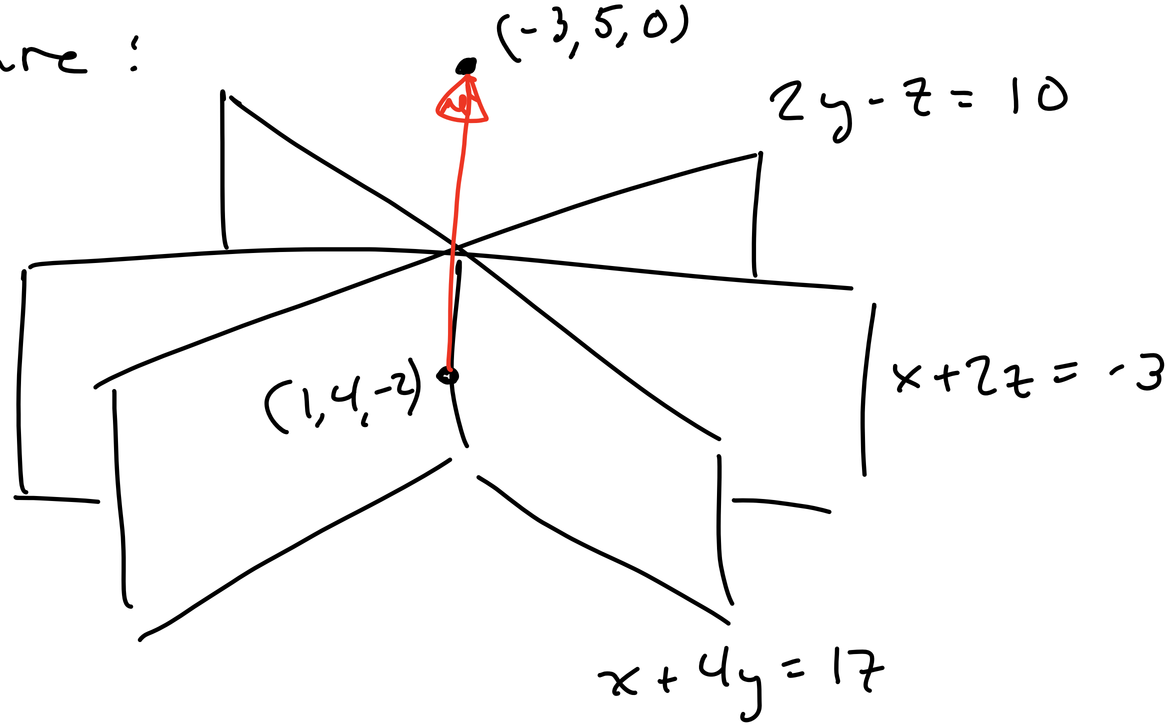
(iii) $(y-4)/1 = (z+2)/2$

$$2(y-4) = z+2$$

$$2y - 8 = z + 2$$

$$2y - z = 10$$

Picture:



Conversely, suppose we are given two planes. Find a parametrization for the line of intersection.

$$\begin{aligned} &① \quad ax + by + cz = d \\ &② \quad Ax + By + Cz = D \end{aligned}$$

$\rightsquigarrow \quad \vec{r}(t) = (x_0 + tu, \, y_0 + tv, \, z_0 + tw)$

Example:

$$\begin{aligned} &① \quad x + y + z = 4, \\ &② \quad x + 2y + 3z = 3. \end{aligned}$$

We'll use the method of "elimination".

First subtract equations to eliminate x:

$$
\begin{array}{rl}
 & (\cancel{x} + 2y + 3z = 3) \\
- & (\cancel{x} + y + z = 4) \\
\hline
③ & y + 2z = -1
\end{array}
$$

Get new equation ③ with no x.

This gives a simpler, but equivalent, system of equations:

$$
\begin{array}{l}
① \\
③
\end{array}
\begin{cases}
x + y + z = 4, \\
\quad\; y + 2z = -1.
\end{cases}
$$

Finally, we use eq ③ to eliminate y from eq ①. Take ① − ③

$$
\begin{array}{rl}
 & (x + \cancel{y} + z = 4) \\
- & (0 + \cancel{y} + 2z = -1) \\
\hline
\end{array}
$$

$$④ \quad x + 0 - z = 5$$

Our final equivalent system is

$$\begin{cases} ④ \quad \boxed{x + 0} - z = 5, \\ ③ \quad \boxed{0 + y} + 2z = -1. \end{cases}$$

The good thing: We have "solved" for the "pivot variables" x & y, in terms of the "free variable" z.

Let's write down the solution

$$\begin{cases} x = 5 + z \\ y = -1 - 2z. \end{cases}$$

This looks like a parametrized line with parameter z.

WEIRD: Let's define $t = z$.

Then we really do get a parametrized line:

$$\begin{cases} x = 5 + t \\ y = -1 - 2t \\ z = t \end{cases}$$

$$\vec{r}(t) = (x(t), y(t), z(t))$$

$$= (5 + t, -1 - 2t, 0 + t)$$

$$(5, -1, 0) + t\,(1, -2, 1)$$

Picture: