Chap 1: Parametrized paths & Arc length

Now: Chapter 2, vectors, equations of lines & planes.

Equation of a line can be written in several different ways:

slope, intercept $\quad y = mx + b$

slope, point $\quad m = (y - y_0)/(x - x_0)$

two points (x_0, y_0) & (x_1, y_1) $\quad m = (y_2 - y_1)/(x_2 - x_1)$

$$\frac{(y - y_1)}{(x - x_1)} = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$(y - y_1)(x_2 - x_1) = (x - x_1)(y_2 - y_1)$$

In this course we use the form

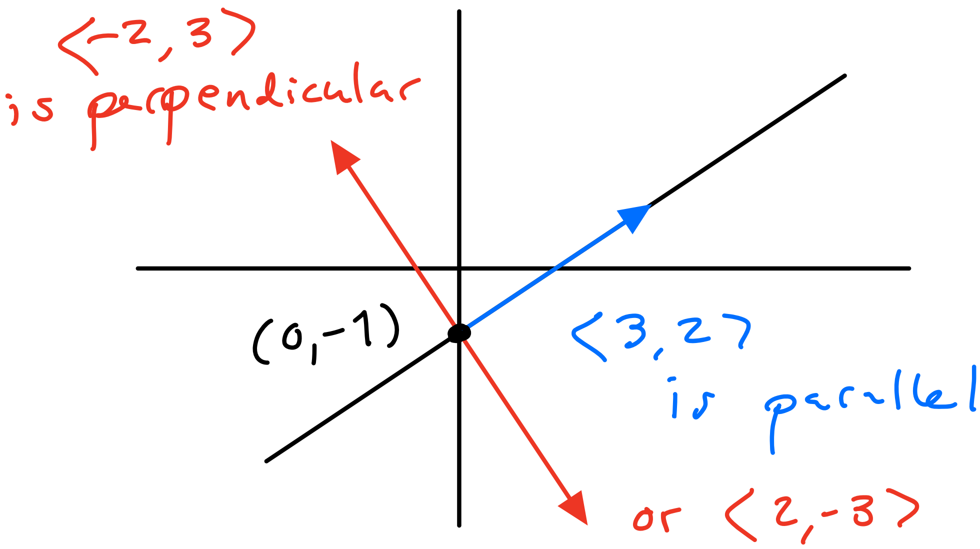
$$ax + by = c$$

OR

$$a(x - x_0) + b(y - y_0) = 0$$

This is the line passing through point (x_0, y_0) & $\perp$ to vector $\langle a, b\rangle$.

Example: $y = \frac{2}{3}x - 1$



Convert to point, perpendicular form.

Take $(x_0, y_0) = (0, -1)$.

Need a vector .....

Trick: Find a vector in the line (say $\langle 3,2\rangle$ because slope $2/3$), then take the negative reciprocal slope $-3/2$, which suggests

$$\langle a, b\rangle = \langle -2, 3\rangle \text{ or } \langle 2, -3\rangle$$

In fact, <u>any</u> scalar multiple of $\langle -2, 3\rangle$ or $\langle 2, -3\rangle$ will work.

Thus we get the equation

$$a(x - x_0) + b(y - y_0) = 0$$

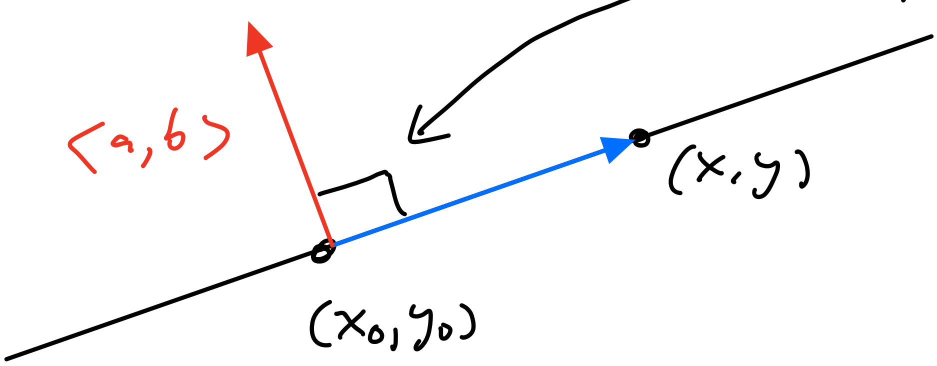
$$2(x-0) - 3(y+1) = 0.$$

$\langle a, b\rangle = \langle 2, -3\rangle$ $\qquad$ $(x_0, y_0) = (0, -1).$

This equation says that

$$\langle a, b\rangle \bullet \langle x - x_0, y - y_0\rangle = 0$$

vectors are perp.



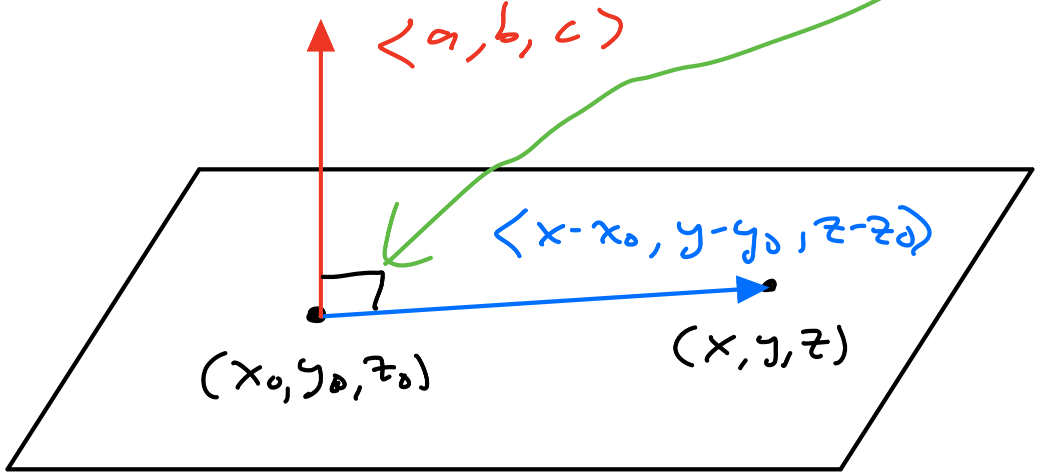
More into 3D:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0.$$

$$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0.$$

perpendicular

Picture: A plane. Given point (x_0,y_0,z_0) in the plane & vector $\langle a,b,c\rangle$ $\perp$ to the plane, any point (x,y,z) in plane must satisfy this equation.



[Perp vector $\langle a,b,c\rangle$ is also called a "normal vector" for the plane, so

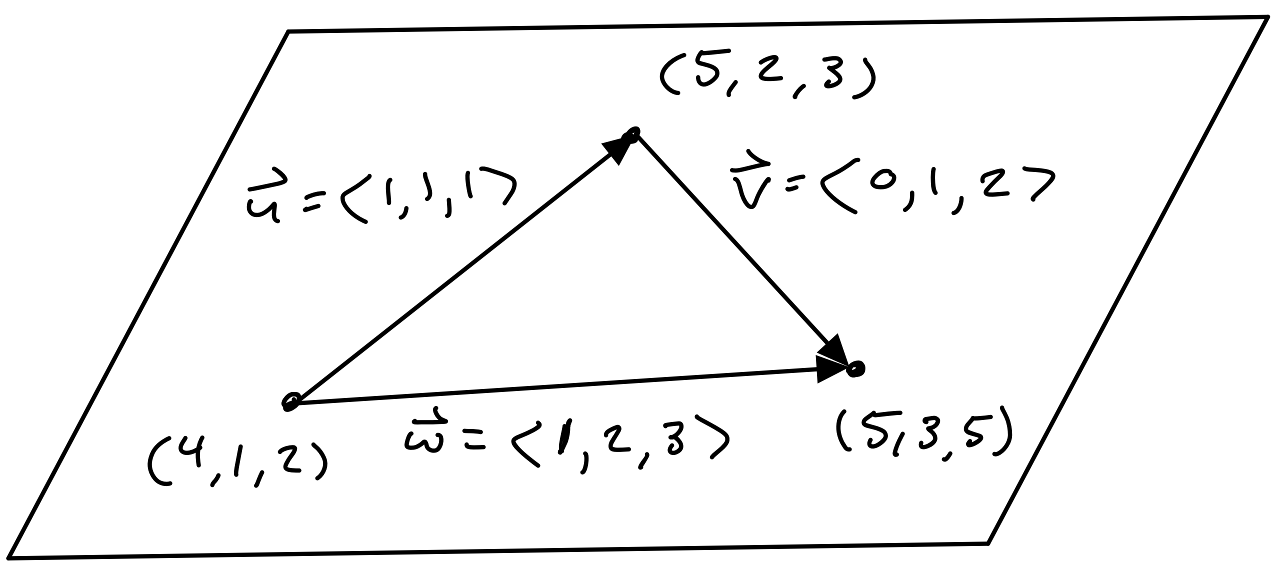
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

is the "normal equation."]

Example: 3 points determine a plane. Find the normal equation of the plane containing points

$$P=(4,1,2)$$
$$Q=(5,2,3)$$
$$R=(5,3,5)$$

Triangle in space



We need to find a normal vector

$\vec{n} = \langle a, b, c \rangle$

pointing $\perp$ to the plane.

There is a TRICK.

Definition of "Cross Product":

Given two vectors in 3D space

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

we define a new vector $\vec{u} \times \vec{v}$ as follows:

$$\vec{u} \times \vec{v} =$$

$$\langle u_2 v_3 - u_3 v_2 \,,\, u_3 v_1 - u_1 v_3 \,,\, u_1 v_2 - u_2 v_1 \rangle$$

Wow, what a mess!

Two Issues:

(1) What does it mean?

(2) How can we memorize the formula?

(1) Meaning: $\vec{u} \times \vec{v}$ is simultaneously $\perp$ to $\vec{u}$ & $\vec{v}$.

Check:

$\vec{u} \cdot (\vec{u} \times \vec{v}) =$

$u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1)$

$= \cancel{u_1 u_2 v_3} + \cancel{u_2 u_3 v_1} + \cancel{u_3 u_1 v_2}$
$\quad - \cancel{u_1 u_3 v_2} - \cancel{u_2 u_1 v_3} - \cancel{u_3 u_2 v_1}$

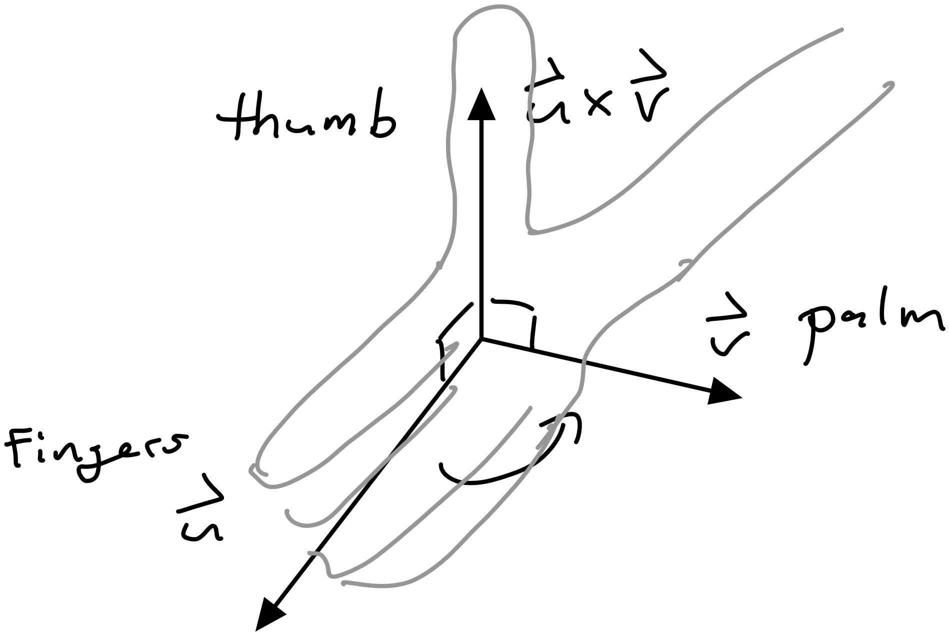
$= 0$ Magic!

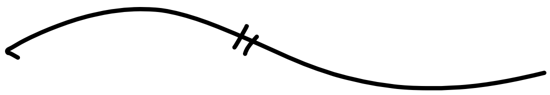
So $\vec{u} \perp \vec{u} \times \vec{v}$.

Similar computation shows $\vec{v} \perp \vec{u} \times \vec{v}$.

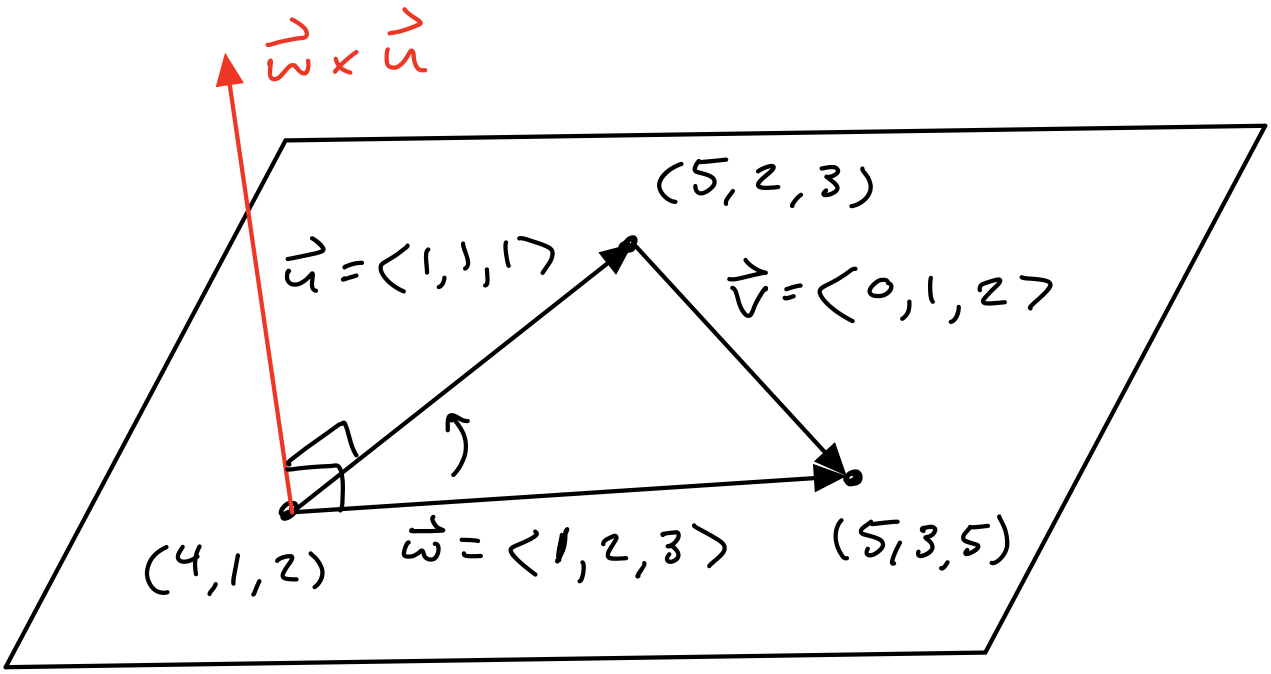
[The 3D version of the "negative reciprocal slope" trick.]

Picture: Right Hand Rule.





Back to our example:



We have 6 different ways to create a normal vector

$\vec{u}\times\vec{v}$, $\vec{u}\times\vec{w}$, $\vec{v}\times\vec{w}$

$\vec{v}\times\vec{u}$, $\vec{w}\times\vec{u}$, $\vec{w}\times\vec{v}$.

Let's do this one

$$\vec{w} = \langle 1, 2, 3\rangle$$

$$\vec{u} = \langle 1, 1, 1\rangle$$

$$\vec{w}\times\vec{u} = \langle 1, 2, 3\rangle \times \langle 1, 1, 1\rangle$$

$$= \langle 2\cdot 1 - 3\cdot 1, 3\cdot 1 - 1\cdot 1, 1\cdot 1 - 2\cdot 1\rangle$$

$$= \langle -1, 2, -1\rangle.$$

Check:

$$\langle 1,2,3\rangle \cdot \langle -1,2,-1\rangle = -1+4-3 = 0 \checkmark$$

$$\langle 1,1,1\rangle \cdot \langle -1,2,-1\rangle = -1+2-1 = 0 \checkmark$$

Finally we get the normal equation of the plane. Take

$$\langle a,b,c\rangle = \langle -1, 2, -1\rangle \leftarrow$$ we had 6 choices

$(x_0, y_0, z_0) = (4, 1, 2) \leftarrow$ we had 3 choices

Equation :

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-1(x-4) + 2(y-1) - 1(z-2) = 0$$

simplify if we want

$$-x + 2y - z + 4 \cancel{-2} + \cancel{2} = 0$$

$$-x + 2y - z = -4$$

$$1x - 2y + 1z = 4$$

That looks better.

$\langle 1, -2, 1 \rangle$

Normal vector is still visible.

But the point is hidden.

② How can we memorize the cross product ?

Most common mnemonic. Let

$$\left.\begin{array}{l} \vec{i} = \langle 1,0,0\rangle \\ \vec{j} = \langle 0,1,0\rangle \\ \vec{k} = \langle 0,0,1\rangle \end{array}\right\} \text{standard basis vectors}$$

so general vector is

$$\langle a,b,c\rangle = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$$
$$= a\vec{i} + b\vec{j} + c\vec{k}$$

Mnemonic

$\vec{u} = \langle u_1, u_2, u_3\rangle$

$\vec{v} = \langle v_1, v_2, v_3\rangle$

$$\vec{u}\times\vec{v} = \text{``}\begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 & u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \end{matrix}\text{''}$$

$$= +u_2v_3\vec{i} + u_3v_1\vec{j} + u_1v_2\vec{k}$$
$$-u_2v_1\vec{k} - u_3v_2\vec{i} - u_1v_3\vec{j}$$

Example:

$\langle 1,2,3\rangle \times \langle 1,1,1\rangle$

$$= \text{“}\begin{array}{cccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\text{”}$$

$$= +2\vec{i} + 3\vec{j} + 1\vec{k}$$
$$\quad -2\vec{k} - 3\vec{i} - 1\vec{j}$$

$$= (2-3)\vec{i} + (3-1)\vec{j} + (1-2)\vec{k}$$

$$= -1\vec{i} + 2\vec{j} - 1\vec{k}$$

$$= \langle -1, 2, -1\rangle \quad \checkmark$$

Deeper: This trick is based on the concept of a "determinant" of a square matrix, which comes from linear algebra.

2×2 case:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= +aei + bfg + cdh$$
$$-ceg - bdi - afh$$

Recursive Expansion:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Laplace expansion along first row a, b, c

$$= a \det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$$

$$- b \det\begin{pmatrix} d & f \\ g & i \end{pmatrix}$$

$$+ c \det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Cross product:

$$\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle$$

$$= \text{``}\det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}\text{''}$$

Not literal because $\vec{i}, \vec{j}, \vec{k}$ are vectors not scalars. But whatever.

Ultimate meaning of determinants.

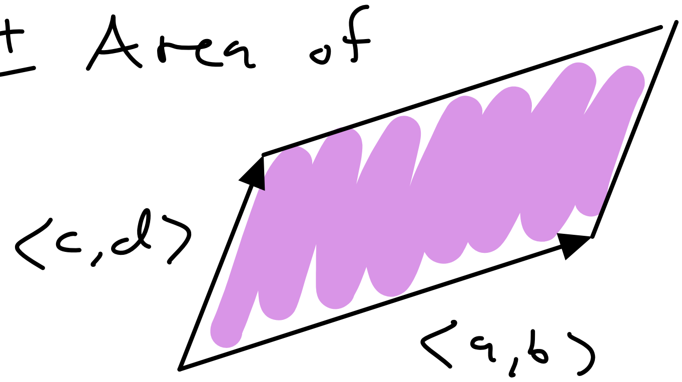
Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ be n vectors in n-dimensional space.

$$\det \begin{pmatrix} — \vec{u}_1 — \\ — \vec{u}_2 — \\ \vdots \\ — \vec{u}_n — \end{pmatrix} = \pm \text{ Volume of parallelepided generated by } \vec{u}_1, \dots, \vec{u}_n.$$

$n \times n$ matrix

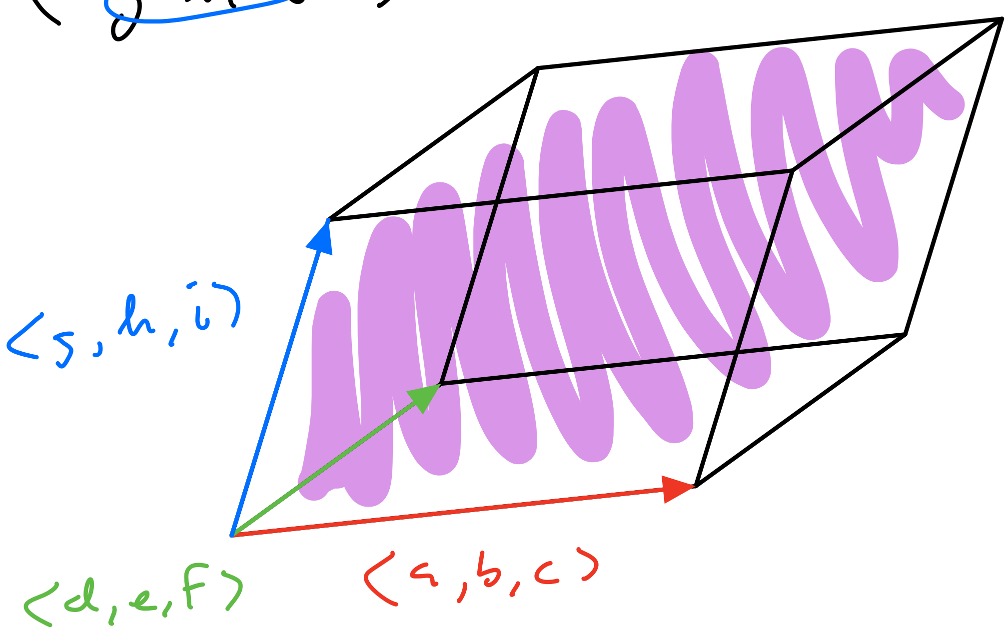
2×2 :

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \text{ Area of}$$



3×3 :

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \pm \text{ Volume of}$$



[This is behind "u-substitution" for 2D & 3D integration, as we'll see.]

Algebraic Rules for Cross Product (Only applies in 3D)

- $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- $a(\vec{u} \times \vec{v}) = (a\vec{u}) \times \vec{v} = \vec{u} \times (a\vec{v})$
- $\vec{u} \times \vec{0} = \vec{0}$
- $\vec{u} \times \vec{u} = \vec{0}$

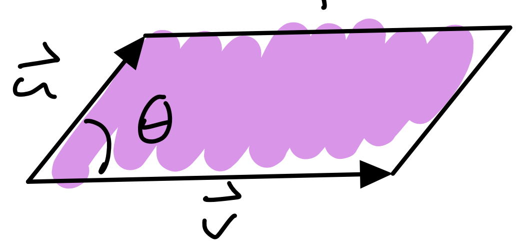
the zero vector, not the zero scalar.

- $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$
- $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$

the scalar zero.

- $\underbrace{|\vec{u} \times \vec{v}|}_{\text{length of the vector } \vec{u} \times \vec{v}} = \|\vec{u}\| \|\vec{v}\| \sin\theta$

$= \pm$ area of the parallelogram



- $\vec{u} \cdot (\vec{v} \times \vec{w}) = \pm$ volume of parallelepided spanned by $\vec{u}, \vec{v}, \vec{w}$.

Reason: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{pmatrix} - & \vec{u} & - \\ - & \vec{v} & - \\ - & \vec{w} & - \end{pmatrix}$.