HW 1 will be posted today, due Fri.

Recall:

$\mathbb{R}$ = the set of real numbers

= the number line.

$\mathbb{R}^2$ = ordered pairs of real numbers

= the coordinate plane.

$\vdots$

$\mathbb{R}^n$ = ordered $n$-tuples of real numbers

= coordinate "$n$-space"

A function $f: \mathbb{R} \to \mathbb{R}^2$ can be thought of as a parametrized curve in the plane. Let's write

$$\underset{\substack{\uparrow \\ \text{input} \\ \text{from } \mathbb{R}}}{f(t)} = (x(t), y(t))$$

output in the plane $\mathbb{R}^2$

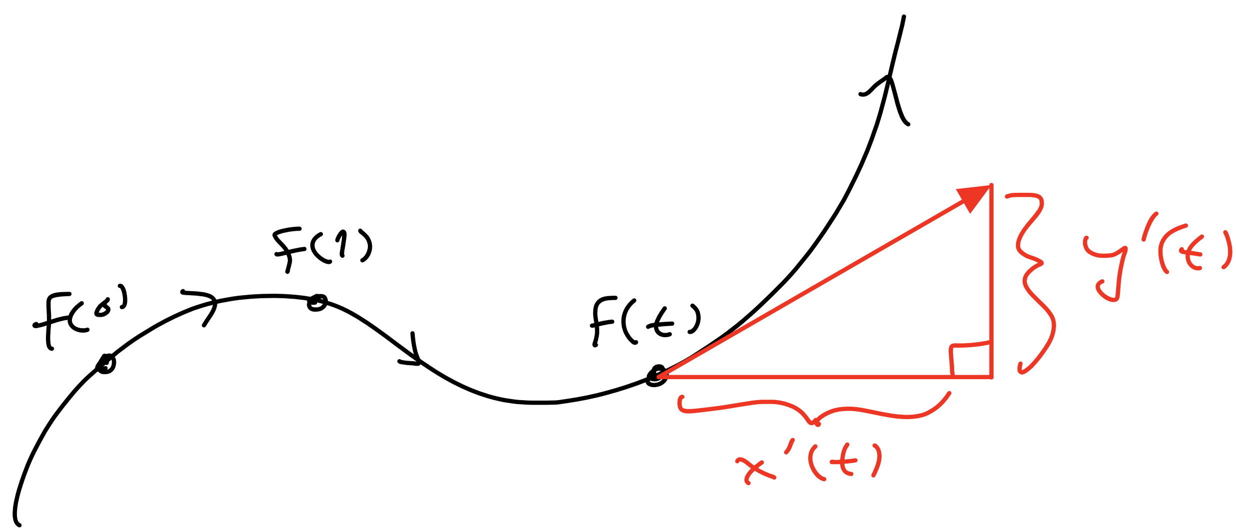
The derivative $f'(t)$ or $df/dt$ is defined as

$$f'(t) = (x'(t), y'(t))$$

$$\frac{df}{dt} = \left( \frac{dx}{dt}, \frac{dy}{dt} \right)$$

Note that $f' : \mathbb{R} \to \mathbb{R}^2$.

Picture:

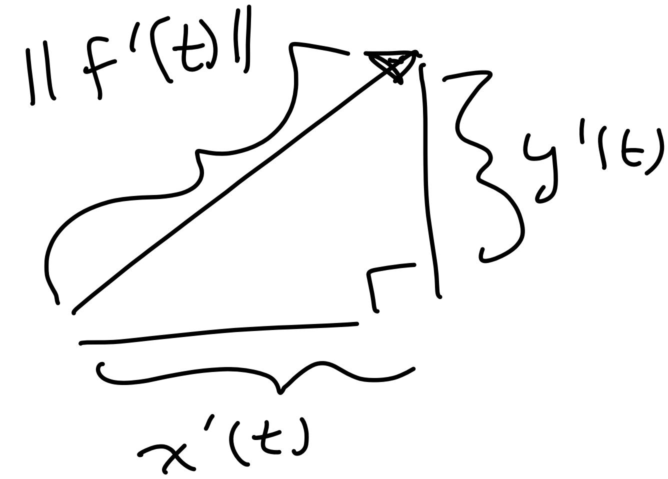



Every vector has a length or magnitude. The magnitude of velocity is speed.

$$\text{speed}(t) = \| f'(t) \|$$

$$= \sqrt{x'(t)^2 + y'(t)^2}.$$

Picture: Pythagorean Theorem.




$$\|f'(t)\|^2 = x'(t)^2 + y'(t)^2$$

Just as

$$\text{speed} = \frac{d}{dt}\,\text{distance},$$

we have

$$\text{distance} = \int \text{speed}\, dt$$

Arc length of path $f(t) = (x(t), y(t))$

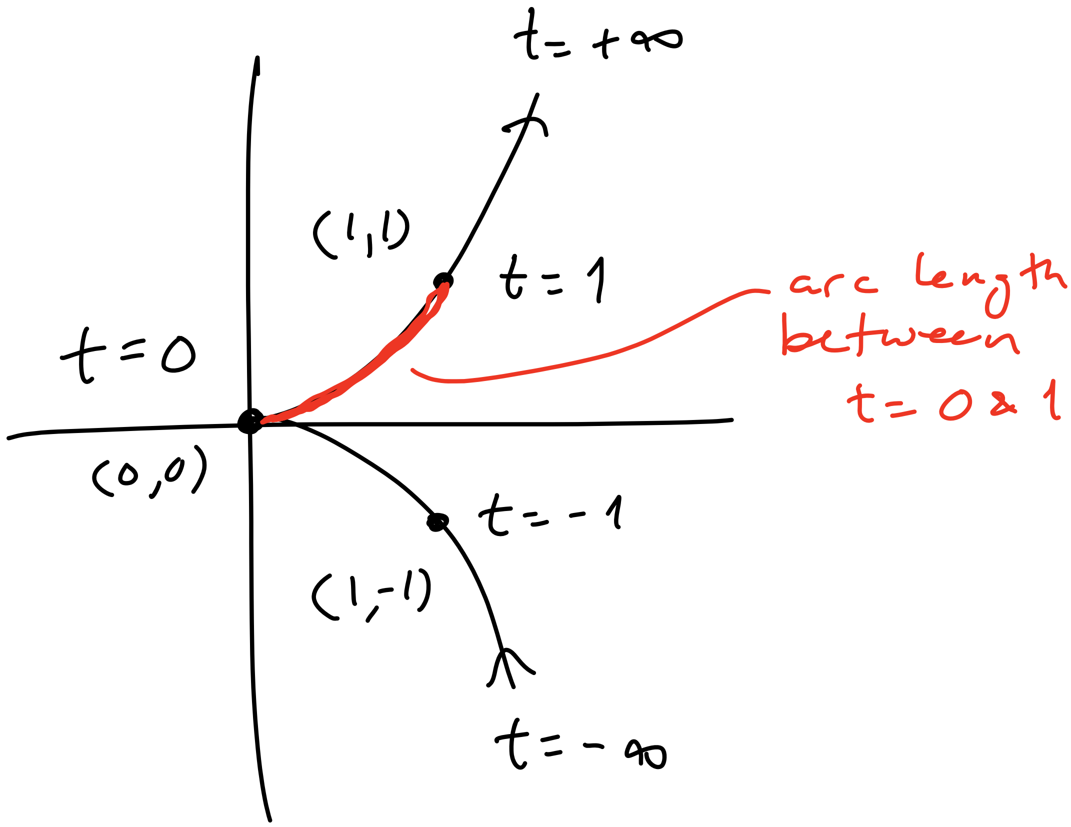
$$= \int \text{speed}\, dt = \int \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Generally, arc length integrals are impossible to solve by hand.

Here's a peculiar curve whose arc length is solvable by hand.

William Neile (1657):

$$f(t) = (t^2, t^3).$$




velocity $f'(t) = (2t, 3t^2)$

At time $t = 0$, velocity becomes

$(0,0)$. The particle briefly stops, then changes direction.

$$\text{speed} = \|\vec{r}\,'(t)\|$$
$$= \sqrt{(2t)^2 + (3t^2)^2}$$
$$= \sqrt{4t^2 + 9t^4}$$
$$= \sqrt{t^2(4+9t^2)}$$
$$= |t|\sqrt{4+9t^2}$$

Luckily this function can be integrated by hand.

Arc length

$$\int_{t=0}^{t=1} t\sqrt{4+9t^2}\,dt$$

$$\left[\, u = 4+9t^2, \quad du = 18\,t\,dt \,\right]$$

$$= \int_{u=4}^{u=13} \sqrt{u} \cdot \frac{du}{18} \qquad \int u^{1/2} = \frac{u^{3/2}}{3/2}$$

$$= \frac{1}{18} \cdot \frac{u^{3/2}}{3/2} \Bigg|_{u=4}^{u=13} \qquad \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{18}_{9}}$$

$$= \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

$$\approx 1.439$$

It doesn't look nice but we did it by hand!

Chapter 2: Vectors

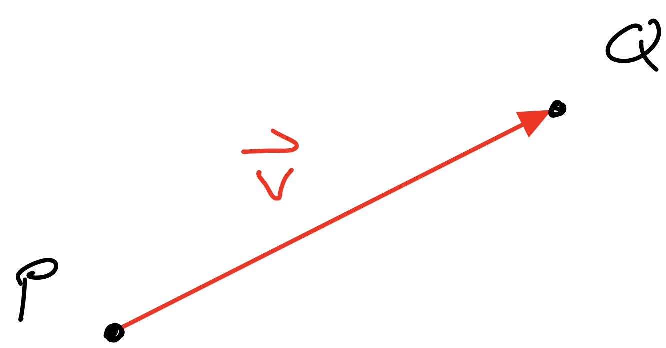
What is a "vector"?

- Physics: A vector is a

"quantity" with direction and magnitude.

- In this class, a vector is a directed line segment (an "arrow") in $\mathbb{R}^2$ or in $\mathbb{R}^3$.

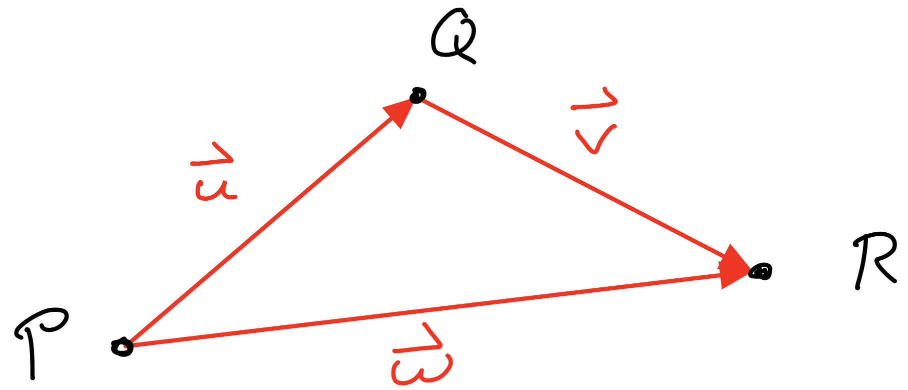
A vector is determined by an ordered pair of points $P$ & $Q$, called the "tail" & "head":




Notation: $\vec{v} = \overrightarrow{PQ}$

There is an "arithmetic" of vectors. They can be <u>added</u> and <u>scaled by constants</u>.

Vectors are added "head-to-tail"




$$\vec{u} + \vec{v} = \vec{w}$$

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

But why do we call this "addition"?

Definition of "coordinates" (or "components" of a vector:

If $P = (x_1, y_1)$ & $Q = (x_2, y_2)$

then we write

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

emphasize that we are talking about a vector not a point.

Then addition of vectors becomes addition of components.

$$P = (x_1, y_1)$$
$$Q = (x_2, y_2)$$
$$R = (x_3, y_3)$$

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$$
$$\overrightarrow{QR} = \langle x_3 - x_2, y_3 - y_2 \rangle$$
$$\overrightarrow{PR} = \langle x_3 - x_1, y_3 - y_1 \rangle$$

Add component by component:

$$\overrightarrow{PQ} + \overrightarrow{QR} =$$
$$= \left\langle (\cancel{x_2} - x_1) + (x_3 - \cancel{x_2}), (\cancel{y_2} - y_1) + (y_3 - \cancel{y_2}) \right\rangle$$
$$= \langle x_3 - x_1, y_3 - y_1 \rangle$$
$$= \overrightarrow{PR} \quad \checkmark$$

Example: $P = (1, 1)$, $Q = (3, 5)$, $R = (6, 3)$.

$\vec{u} = \overrightarrow{PQ} = \langle 3-1, 5-1 \rangle = \langle 2, 4 \rangle$

$\vec{v} = \overrightarrow{QR} = \langle 6-3, 3-5 \rangle = \langle 3, -2 \rangle$

$\vec{w} = \overrightarrow{PR} = \langle 6-1, 3-1 \rangle = \langle 5, 2 \rangle$

Check:

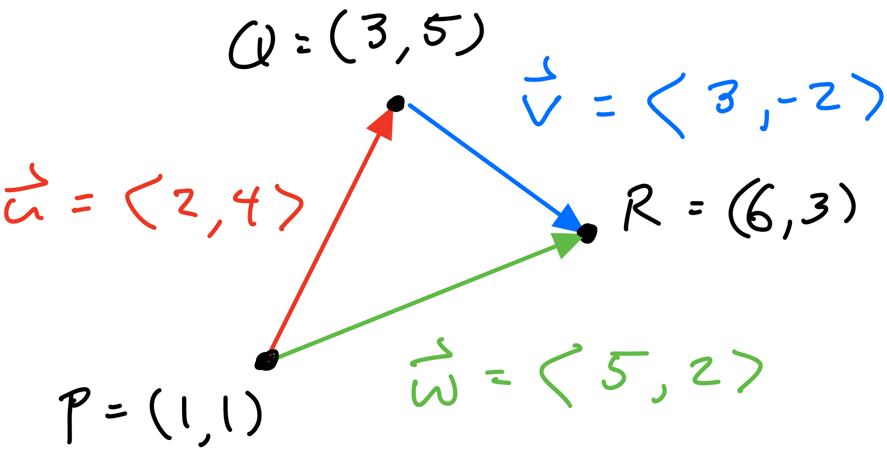
$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

$$\vec{u} + \vec{v} = \vec{w}$$

$$\langle 2, 4 \rangle + \langle 3, -2 \rangle$$

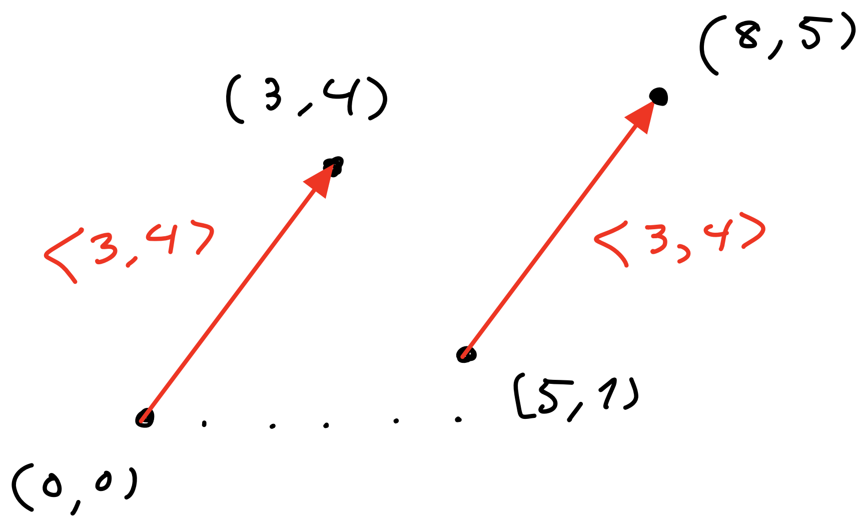
$$= \langle 2+3, 4+(-2) \rangle = \langle 5, 2 \rangle \quad \checkmark$$

Picture:



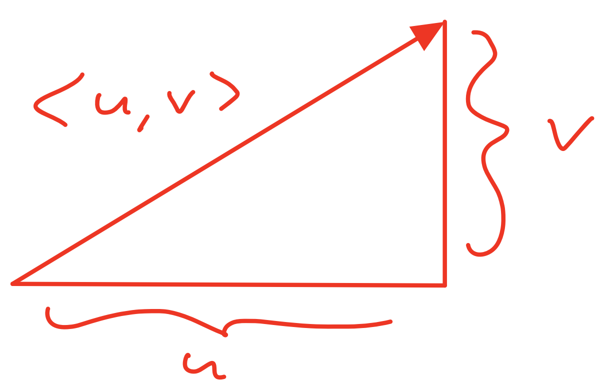
Subtlety: Just as a vector has magnitude & direction, two arrows should be considered "the same" when they have the same magnitude and direction.

e.g.




<u>same vector</u> in different locations.

To compute the magnitude we use the Pythagorean Theorem



$$\| \langle u, v \rangle \|^2 = u^2 + v^2$$

$$\| \langle u, v \rangle \| = \sqrt{u^2 + v^2}$$

e.g. $\| \langle 3, 4 \rangle \| = \sqrt{3^2 + 4^2}$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25} = 5.$$

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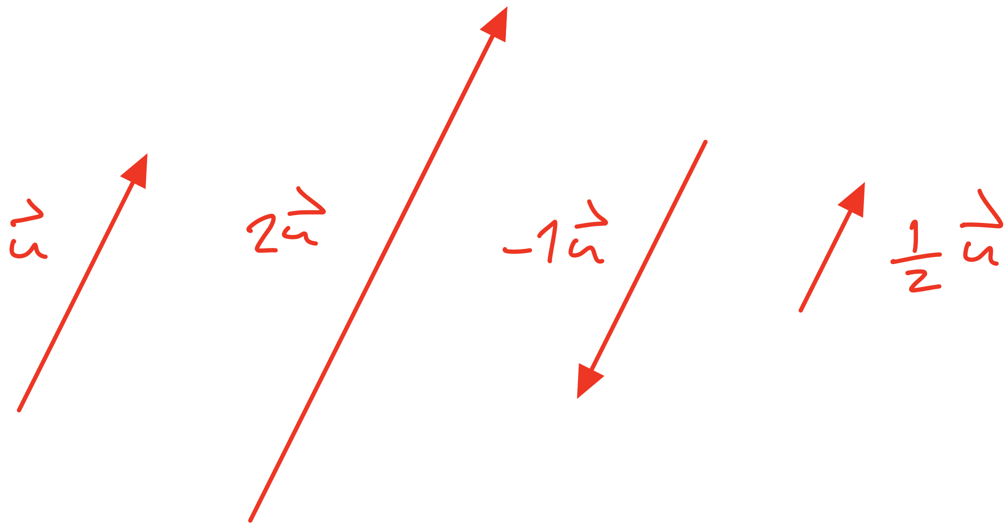
The other vector operation is called "scalar multiplication".

Given vector $\vec{u} = \langle u_1, u_2 \rangle$ and a number (scalar) $k$. we define a new vector $k\vec{u}$ by multiplying each component by $k$:

$$k\vec{u} = \langle ku_1, ku_2 \rangle.$$

This changes the length but not the direction:




Finally, there is a special vector called the "zero vector", all of whose components are zero:

$$\vec{0} = \langle 0, 0 \rangle$$

[ Tail & Head are equal. ]

Here are the rules of vector arithmetic: (page 112 OpenStax)

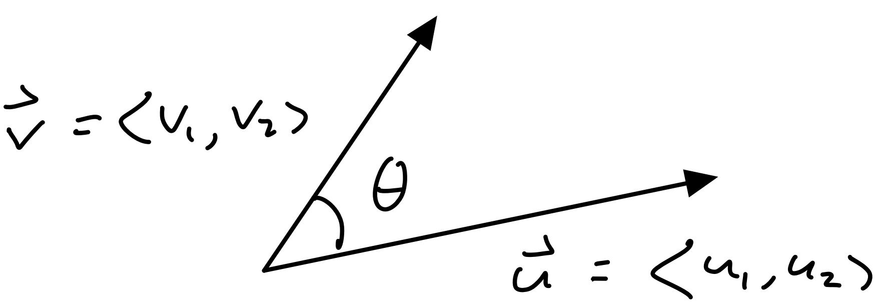
Consider vectors $\vec{u}, \vec{v}, \vec{w}$ and scalars $r, s$.

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

- $\vec{u} + \vec{0} = \vec{u}$
- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(s\vec{u}) = (rs)\vec{u}$
- $(r+s)\vec{u} = r\vec{u} + s\vec{u}$
- $r(\vec{u}+\vec{v}) = r\vec{u} + r\vec{v}$
- $1\vec{u} = \vec{u}$
- $0\vec{u} = \vec{0}$.

Moral: Everything that looks obvious is true!

Vector Arithmetic helps us to solve a hard problem: Find the angle between two vectors.

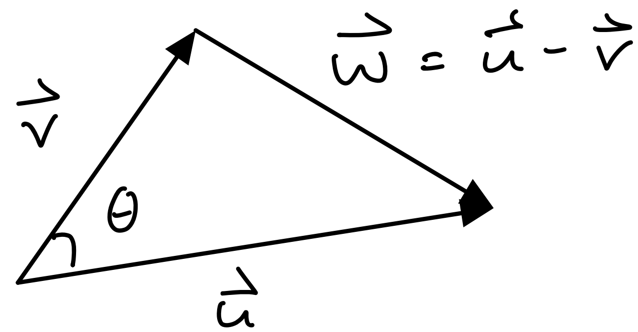


The angle is determined by the four numbers $u_1, u_2, v_1, v_2$ but what is the formula?

$$\theta = \text{some function of } u_1, u_2, v_1, v_2 \;\ldots$$

[The answer will involve a strange arithmetic operation called the "dot product" $\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2$ ]

Key: Draw a triangle:




$$\vec{v} + \vec{w} = \vec{u} \quad \text{so} \quad \vec{w} = \vec{u} - \vec{v}$$

The Law of Cosines says

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

On the other hand, we can compute the lengths in terms of the coordinates $u_1, u_2, v_1, v_2$.

[ Details on HW 1 ]

The result says that

$$\boxed{u_1 v_1 + u_2 v_2 = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

Surprise!

We give this weird expression a name. We define

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 .$$

The Dot Product Theorem says

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta .$$

We can treat this as a third kind of arithmetic operation on vectors. It satisfies some rules:

(pg 147 of OpenStax)

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $s(\vec{u} \cdot \vec{v}) = (s\vec{u}) \cdot \vec{v} = \vec{u} \cdot (s\vec{v})$
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$