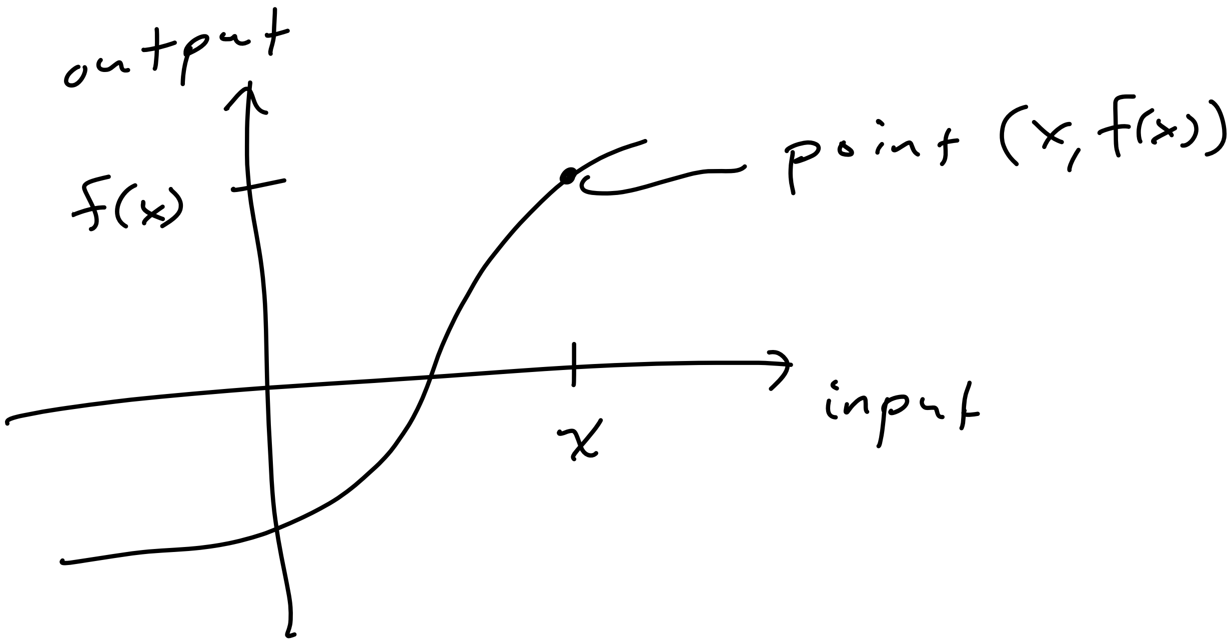
M211 : Calculus 3.

Calc 1 & 2 are based on functions

$$f : \mathbb{R} \to \mathbb{R}$$

one real input — one real output.

Such functions are visualized by considering their "graph":



Calc 3 is based on functions

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

m real inputs — n real outputs

In particular for $m, n = 1, 2, 3$.

This is more relevant to the real world (physics) because the real world is 3D.

Such functions are harder to visualize. We will develop some intuitions:

- $f: \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ is a parametrized path in the plane or space.

- $f: \mathbb{R}^2$ or $\mathbb{R}^3 \longrightarrow \mathbb{R}$ is called a "scalar field". It associates a number (e.g. temperature) to each point in the plane or space.

- $f: \mathbb{R}^2 \to \mathbb{R}^2$
  or $\mathbb{R}^3 \to \mathbb{R}^3$
  are called "vector fields",

e.g. electric field or
gravitational field, . . .

This week: Chapter 2 with a brief look at Chapter 1.

Consider a function $f : \mathbb{R} \to \mathbb{R}^2$.
We will use the notation

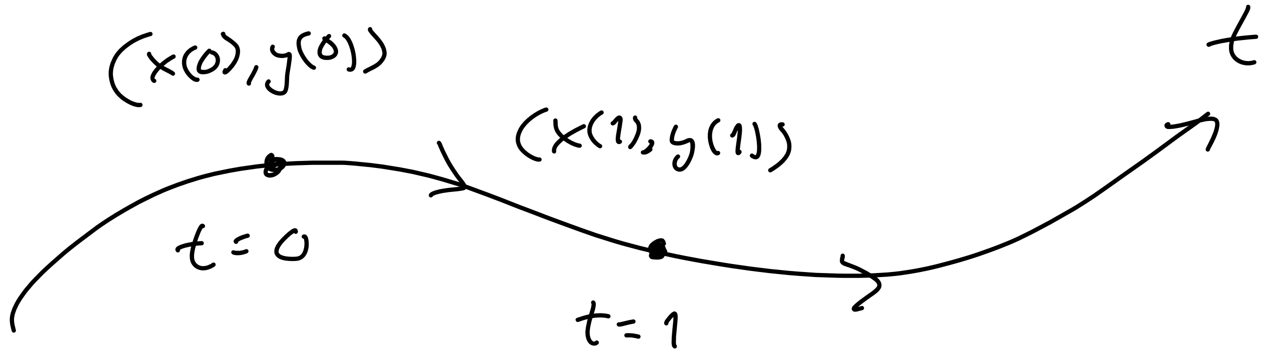
$$f(t) = (x(t), y(t))$$

input called t for "time"

outputs $x(t), y(t)$ are functions of t.

Think: $(x(t), y(t))$ is the position of a moving particle in the real x,y-plane.

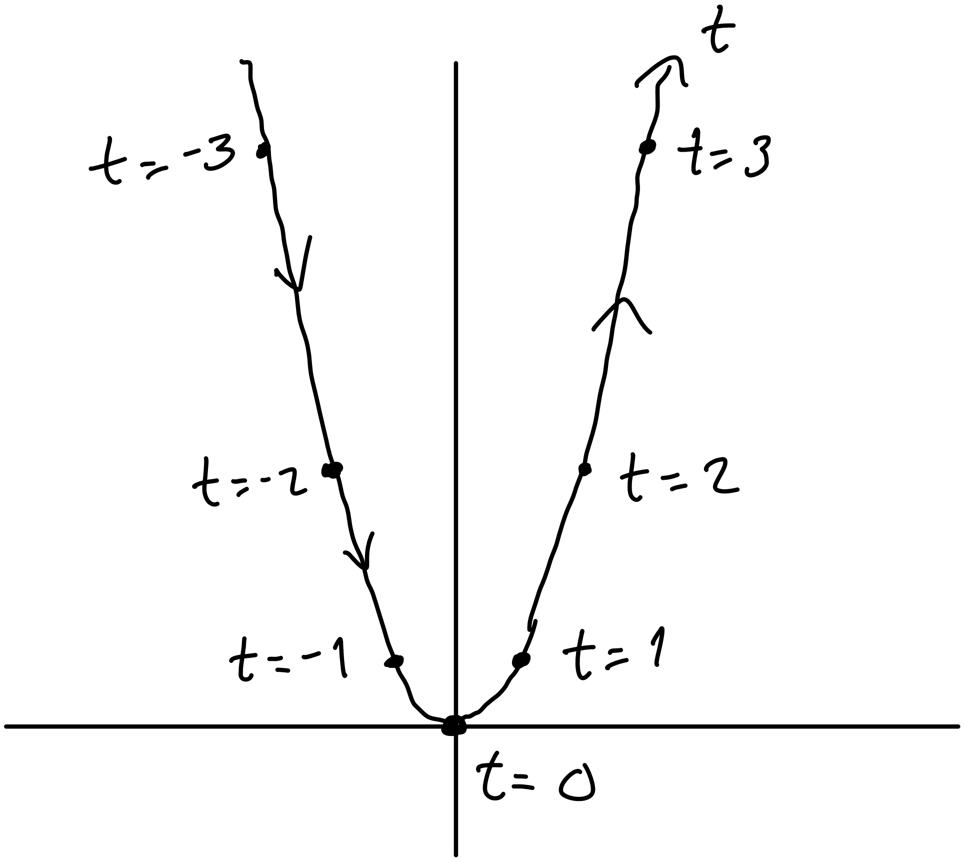
Picture:



Examples: $f(t) = (t, t^2)$.

i.e. let $x(t) = t$ & $y(t) = t^2$.

What does it look like?



It looks like a parabola.

Actually it is a parabola. We can see this by "eliminating t":

$$x = t \rightsquigarrow x^2 = t^2$$
$$y = t^2$$

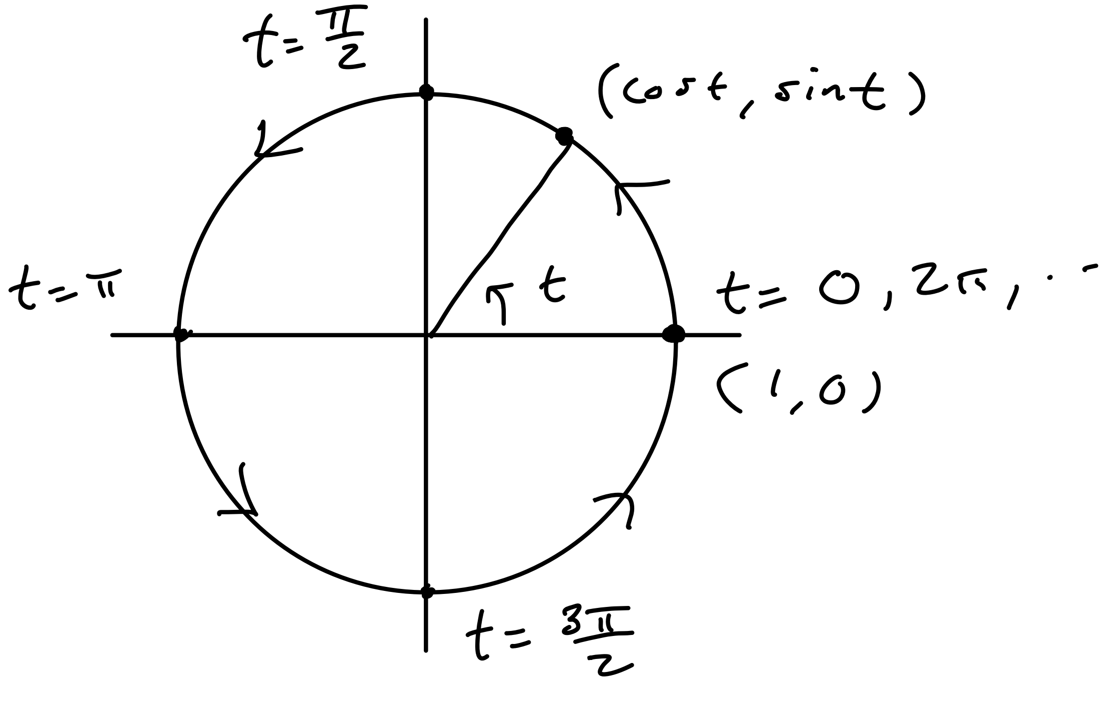
so $y = x^2$ (parabola).

- Let $g(t) = (\cos t, \sin t)$

i.e. $x(t) = \cos t$

$y(t) = \sin t$.

What does it look like?



Path travels the unit circle counter-clockwise, repeats every 2π units of time.

Get the equation of the circle by "eliminating t":

$$x = \cos t$$
$$y = \sin t$$

TRICK!

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

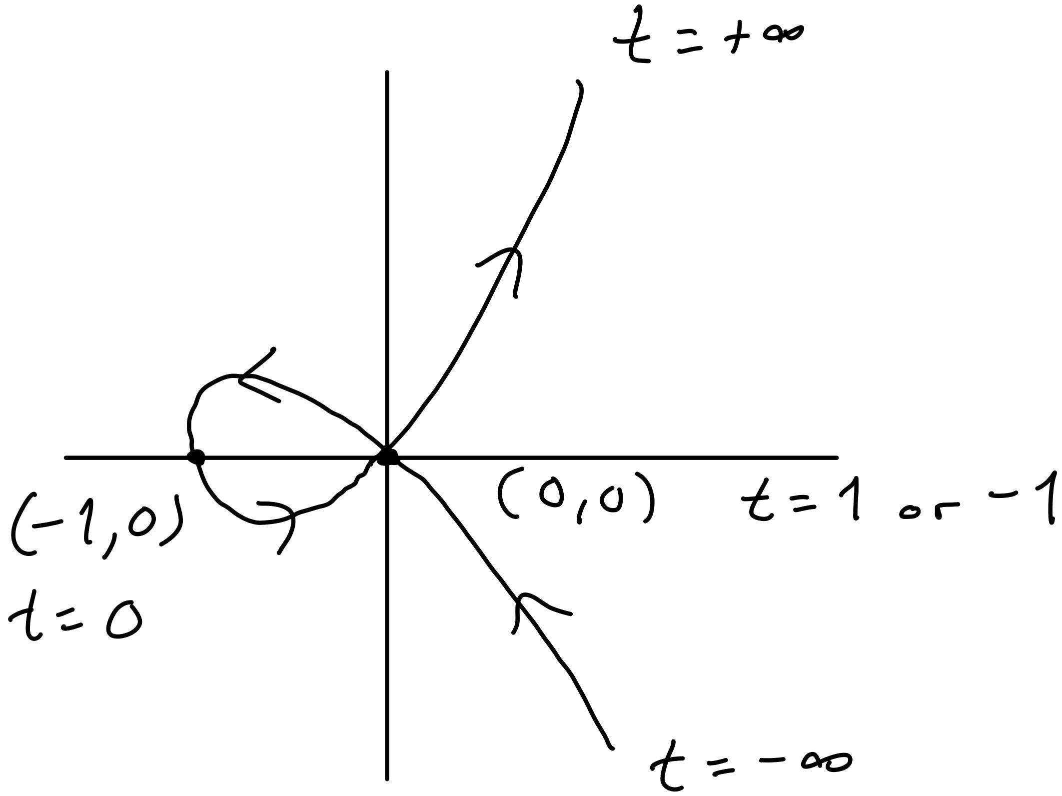
$$x^2 + y^2 = 1$$

(unit circle)

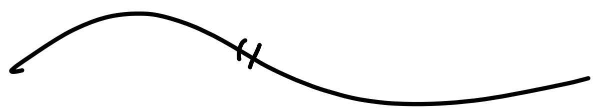
- $h(t) = (t^2 - 1, t^3 - t)$.

What does this look like?

Plot some points:



Interesting: This path intersects itself.



Velocity & Speed.

Given function $f : \mathbb{R} \to \mathbb{R}^2$

written as $f(t) = (x(t), y(t))$

we define its derivative

(with respect to t) as

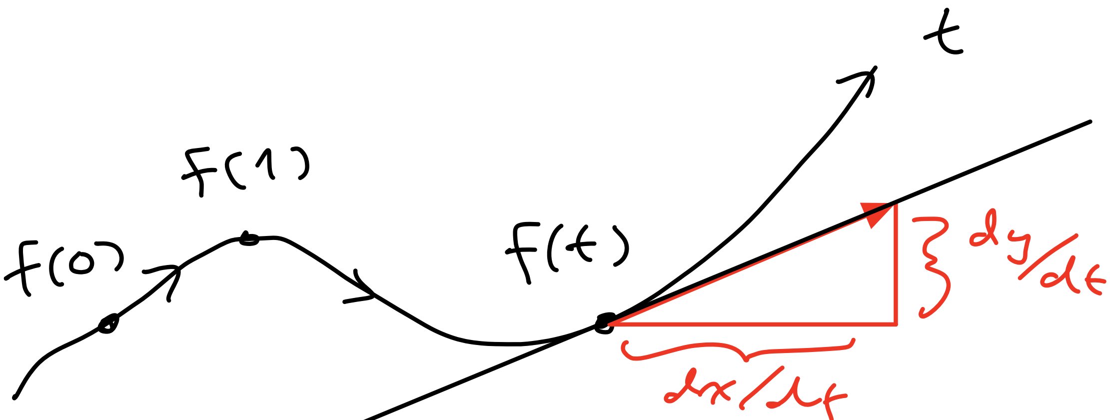
$$F' : \mathbb{R} \to \mathbb{R}^2$$

$$F'(t) = (x'(t), y'(t))$$

$$\frac{dF}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right)$$

Call this the "instantaneous velocity of the parametrized path $f(t)$ at time t.

New Idea: Velocity is a vector.

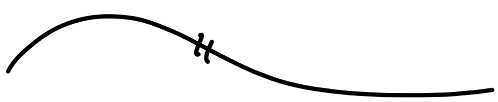


Picture: Velocity is tangent to the path. Hence the slope of the tangent line is

$$\frac{\text{rise}}{\text{run}} = \frac{dy/dt}{dx/dt} = \text{``}\frac{dy}{dx}\text{''}$$

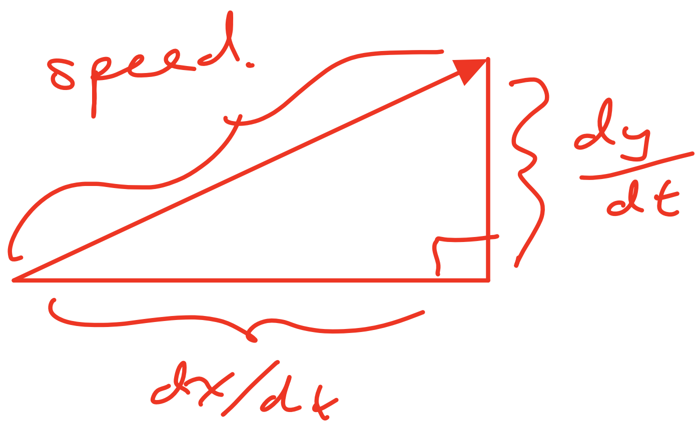
To repeat: Think of $f: \mathbb{R} \to \mathbb{R}^2$ as a parametrized path in $\mathbb{R}^2$.

Think of derivative $f': \mathbb{R} \to \mathbb{R}^2$ as the velocity vectors of the path.



velocity is a vector.

<u>speed</u> is the length or magnitude of the velocity vector:



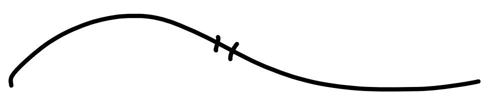
Pythagorean theorem:

$$\text{speed}^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= x'(t)^2 + y'(t)^2$$

$$\text{speed} = \sqrt{x'(t)^2 + y'(t)^2}$$

$=$ "instantaneous speed at time t"



Recall : Suppose your car has speed $s(t)$ at time t. How far do you travel ?

Between times $t = a$ & b your car travels distance

$$\text{distance} = \int_a^b s(t)\,dt$$

[speed = distance / time

$$s(t) = \frac{d}{dt}\text{ distance}.$$

$$\cancel{\int \frac{d}{dt}}(\text{distance}) = \int s(t)\,dt$$

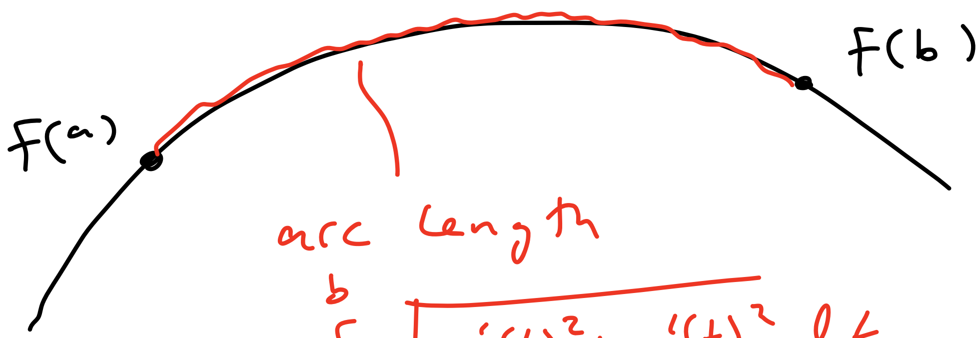
$$\text{distance} = \int \text{speed}$$

The same formula holds in higher dimensions. Given path $f(t) = (x(t), y(t))$, the <u>distance</u> (or "arc length") between $t = a$ & b is

$$\text{distance} = \int_a^b \text{speed}\,dt$$

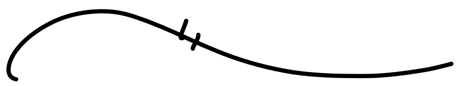
$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

Picture :



arc length

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$$



Examples :

- Parametrized Parabola

$$f(t) = (t, t^2)$$

$$f'(t) = (1, 2t) \quad \text{velocity.}$$

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{1^2 + (2t)^2}$$

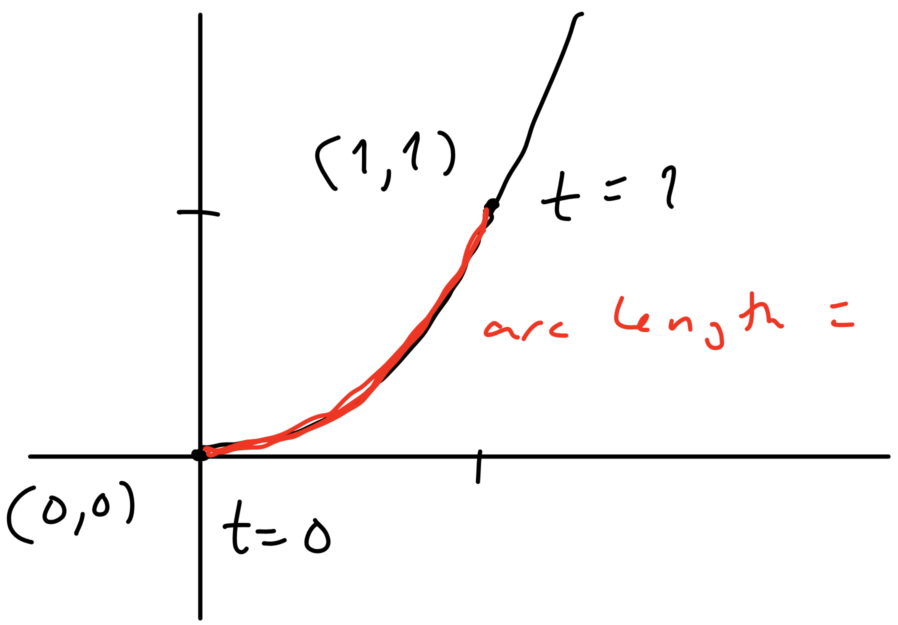
$$= \sqrt{1 + 4t^2} \quad \text{speed.}$$

So the arc length between

times $t = a$ & $t = b$ is

$$\int_a^b \sqrt{1+4t^2}\,dt$$

Say $a = 0$ & $b = 1$.



arc length $= \int_0^1 \sqrt{1+4t^2}\,dt$

Do you know how to compute this?

No. Me neither.

Computer: $\int_0^1 \sqrt{1+4t^2}\,dt \approx 1.479$

[Sadly, most arc length integrals cannot be solved by hand!]

- parametrized unit circle:

$$f(t) = (\cos t, \sin t)$$

$$f'(t) = (-\sin t, \cos t)$$

$$\text{speed} = \sqrt{(-\sin t)^2 + (\cos t)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1} = 1$$

The speed is constant.

So the arc length is easy to compute.

e.g. circumference

$$= \text{arc length between } 0 \ \& \ 2\pi$$

$$= \int_0^{2\pi} 1 \, dt = [t]_0^{2\pi} = 2\pi - 0$$

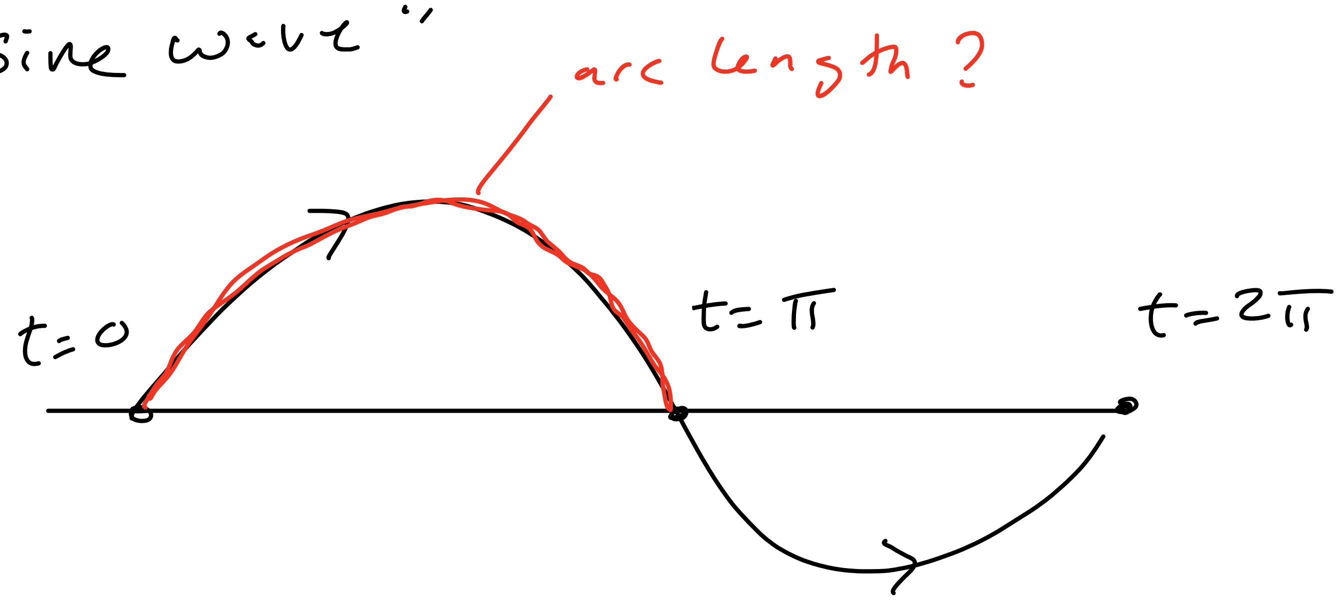
$= 2\pi$.

Yes, this is the circumference of a circle with radius 1. ✓

[HW 1: Do the same thing for a circle of radius r.]

- consider curve $f(t) = (t, \sin t)$.

"sine wave"



velocity $f'(t) = (1, \cos t)$

speed $\sqrt{1^2 + \cos^2 t}$

$$\text{Arc length} = \int_0^{\pi} \sqrt{1+\cos^2 t}\, dt$$

$$\approx 3.82 \text{ (via computer)}$$