HW 4 due tomorrow.

Integration in 3D.

[Chap 5: Integration over 2D regions in $\mathbb{R}^2$ & over 3D regions in $\mathbb{R}^3$.

Chap 6: Integrate along a curve in $\mathbb{R}^2$. Integrate along a curve or surface in $\mathbb{R}^3$.]

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be scalar field

Let $E \subseteq \mathbb{R}^3$ be solid region,

Want to compute

$$\iiint_E \underbrace{f \, \overbrace{dV}^{\text{tiny piece of volume}}}_{?}$$

f = mass density $\rightsquigarrow$ $f\,dV$ = mass.

f = temperature $\rightsquigarrow$ $f\,dV \approx$ heat energy

[Also: $f(x,y,z)$ = "height above" the xyz-space in xyzw-space.
Then $f\,dV$ = 4D hypervolume.]

To compute:

- Pick coordinate system.
- Parametrize region E.

} human

- Compute the integral. ← a computer can do this

Cartesian: $dV = dx\,dy\,dz$.

General coordinates:

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} x(u,v,w) \\ y(u,v,w) \\ z(u,v,w) \end{Bmatrix}$$

Define the Jacobian determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

Volume Forms

$$\underbrace{dx\,dy\,dz}_{\text{tiny volume}} = \underbrace{\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|}_{\substack{\text{volume}\\ \text{stretch factor}}} \underbrace{du\,dv\,dw}_{\text{tiny volume}}$$

e.g. Stretch in 3 directions

$$x = au \qquad y = bv \qquad z = cw$$

constants a, b, c.

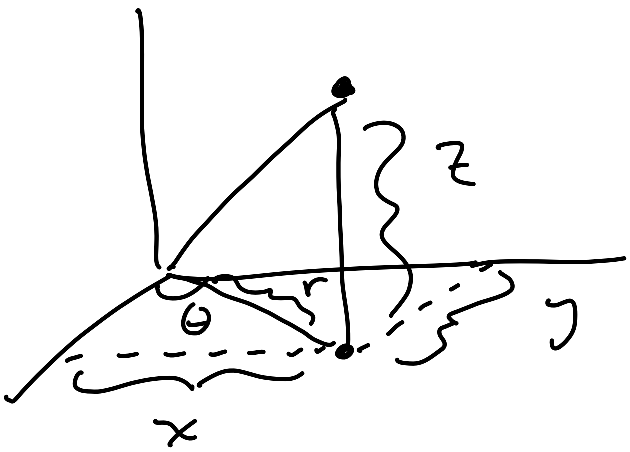
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$$= abc.$$

$$dx\,dy\,dz = \underbrace{abc}_{\substack{\text{volume}\\ \text{stretch factor.}}}\; du\,dv\,dw$$

[See Problem 5 on HW 4.]

e.g. Cylindrical Coords.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

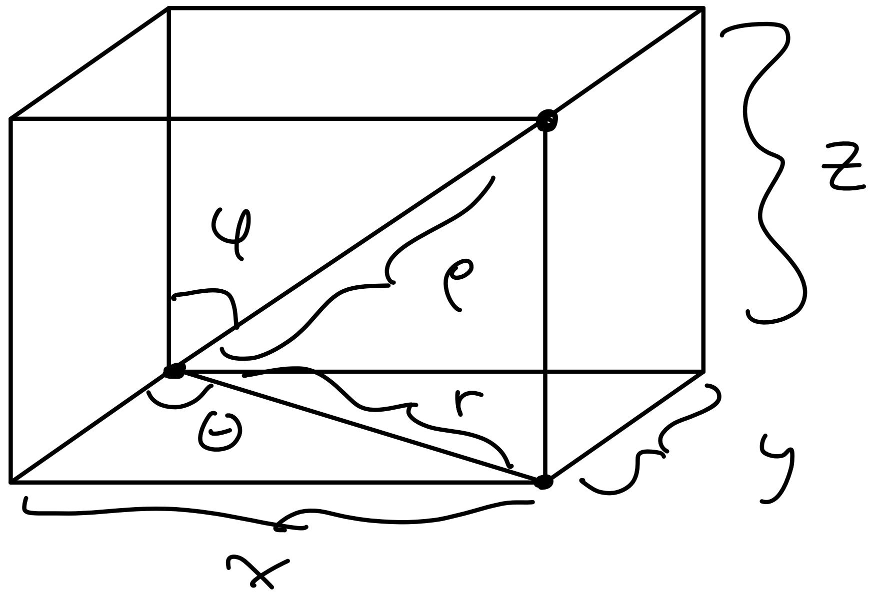
$$= 1 \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r$$

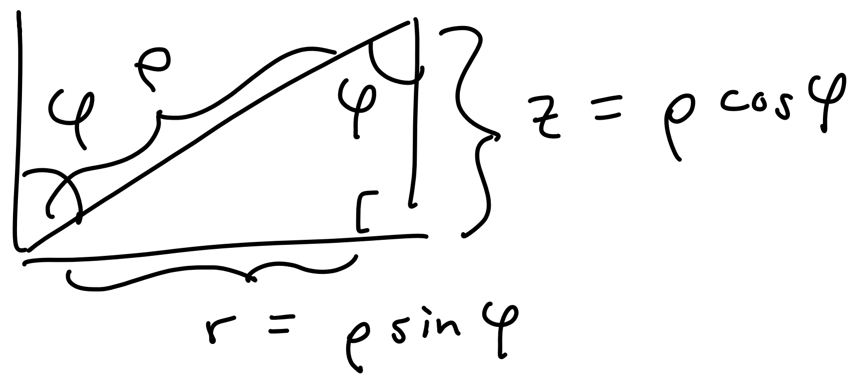
$$\underbrace{dx\,dy}\,dz = r\,\underbrace{dr\,d\theta}\,dz$$

Just polar coords with z attached.

e.g. Spherical Coords



$$x = r\cos\theta \qquad r = \rho\sin\varphi$$
$$y = r\sin\theta \qquad z = \rho\cos\varphi$$
$$r^2 = x^2 + y^2$$



Spherical coords are ρ, θ, φ

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} =$$

$$\det\begin{pmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\rho\sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & 0 \\ \rho\cos\varphi\cos\theta & \rho\cos\varphi\sin\theta & -\rho\sin\varphi \end{pmatrix}$$

$= -\rho^2 \sin\varphi$ (via computer).

$$dx\,dy\,dz = |-\rho^2\sin\varphi|\, d\rho\, d\theta\, d\varphi$$
$$= \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

That's ugly. Let's make sure that it works. [HW 4.5(a)]

Compute volume of sphere of radius a:

$$x^2 + y^2 + z^2 \le a$$

$$\text{volume} = \iiint_{\text{sphere}} 1 \, dV$$

Could use Cartesian coords but the parametrization is a mess:

$$-a \le x \le a$$

$$-\sqrt{a^2 - x^2} \le y \le +\sqrt{a^2 - x^2}$$

$$-\sqrt{a^2 - x^2 - y^2} \le z \le +\sqrt{a^2 - x^2 - y^2}$$

Much better to use spherical coords:

$$0 \le \rho \le a$$

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \pi$$

constant !!

$$\text{volume} = \iiint \underbrace{\rho^2 \sin\varphi}_{\text{separable !!}} \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\varphi \, d\varphi \int_0^{a} \rho^2 \, d\rho$$

$$= 2\pi \left[-\cos\varphi \right]_0^{\pi} \cdot \left[\frac{1}{3}\rho^3 \right]_0^{a}$$

$$= 2\pi \left[\overset{1}{\cancel{-\cos(\pi)}} + \overset{1}{\cos(0)} \right] \cdot \left[\frac{1}{3} a^3 \right]$$

$$= 4\pi \left[\frac{1}{3} a^3 \right]$$

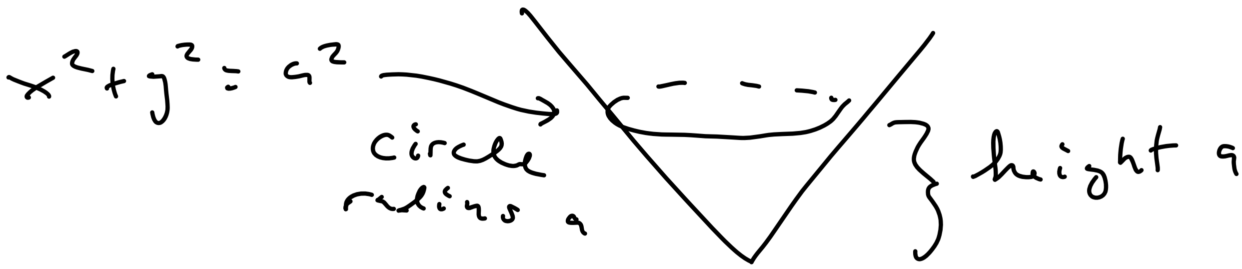
$$= \frac{4}{3}\pi a^3$$

Yes, this is the formula you memorized in 8th grade math.

Harder Example: Find the center of mass of the solid region:

- above the xy-plane
- below the cone $z^2 = x^2 + y^2$
- inside the sphere $x^2 + y^2 + z^2 = 1$.

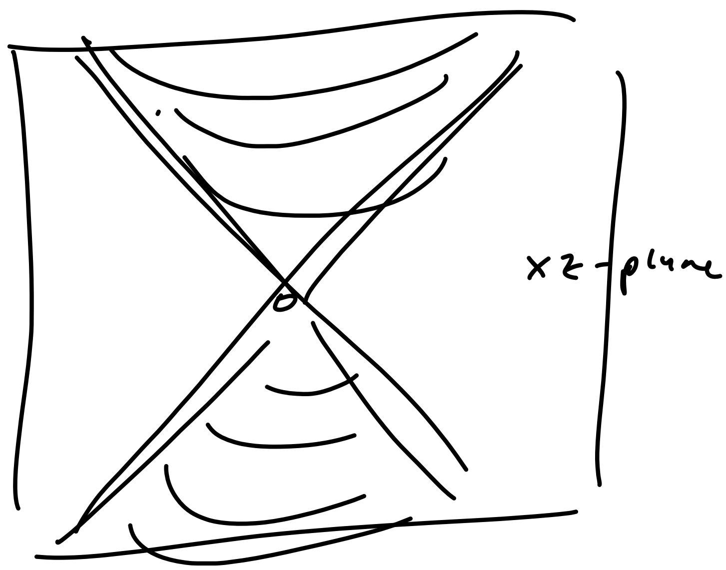
[Why a cone? In the plane $z = a$
The surface is a circle of radius a.



Intersect with plane $y = 0$.

$$x^2 + 0 = z^2$$
$$x^2 - z^2 = 0$$
$$(x - z)(x + z) = 0$$
$$x = \pm z.$$

Two Lines



]

Spherical Coordinates :

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$\frac{\pi}{4} \le \varphi \le \pi/2 \quad \text{(see picture)}$$

Total Mass :

$$m = \iiint 1 \, dV$$

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \cdot \int_{\pi/4}^{\pi/2} \sin\varphi \, d\varphi \cdot \int_0^1 \rho^2 \, d\rho$$

$$= 2\pi \left[-\overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}} + \overset{\sqrt{2}/2}{\cancel{\cos\left(\tfrac{\pi}{4}\right)}} \right] \left[\frac{1}{3} \right]$$

$$= \frac{\cancel{2}\pi}{3} \left[\frac{\sqrt{2}}{\cancel{2}} \right] = \frac{\sqrt{2}}{3}\pi .$$

Center of mass :

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\cancel{\frac{M_{yz}}{m}}, \cancel{\frac{M_{xz}}{m}}, \frac{M_{xy}}{m} \right)$$

$$\cancel{M_{yz} = \iiint x \, dV}$$

zero by symmetry

$$\cancel{M_{xz} = \iiint y \, dV}$$

$$M_{xy} = \iiint z \, dV.$$

$$= \iiint z \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \iiint \rho \cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \iiint \rho^3 \frac{1}{2} \sin(2\varphi) \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi/2} \frac{1}{2} \sin(2\varphi) \, d\varphi \int_0^1 \rho^3 \, d\rho$$

$$= 2\pi \left[-\frac{1}{4} \cos(2\varphi) \right]_{\pi/4}^{\pi/2} \left[\cancel{\frac{1}{4} \rho^4} \right]_0^1$$

$$= \frac{-2\pi}{16}\left[\overset{-1}{\cancel{\cos(\pi)}} - \overset{0}{\cancel{\cos\left(\tfrac{\pi}{2}\right)}}\right]$$

$$= \frac{2\pi}{16} = \frac{\pi}{8}.$$

Finally, the center of mass is:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{\pi/8}{\sqrt{2}\,\pi/3}\right)$$

$$= \left(0, 0, \frac{3}{\sqrt{2}\cdot 8}\right)$$

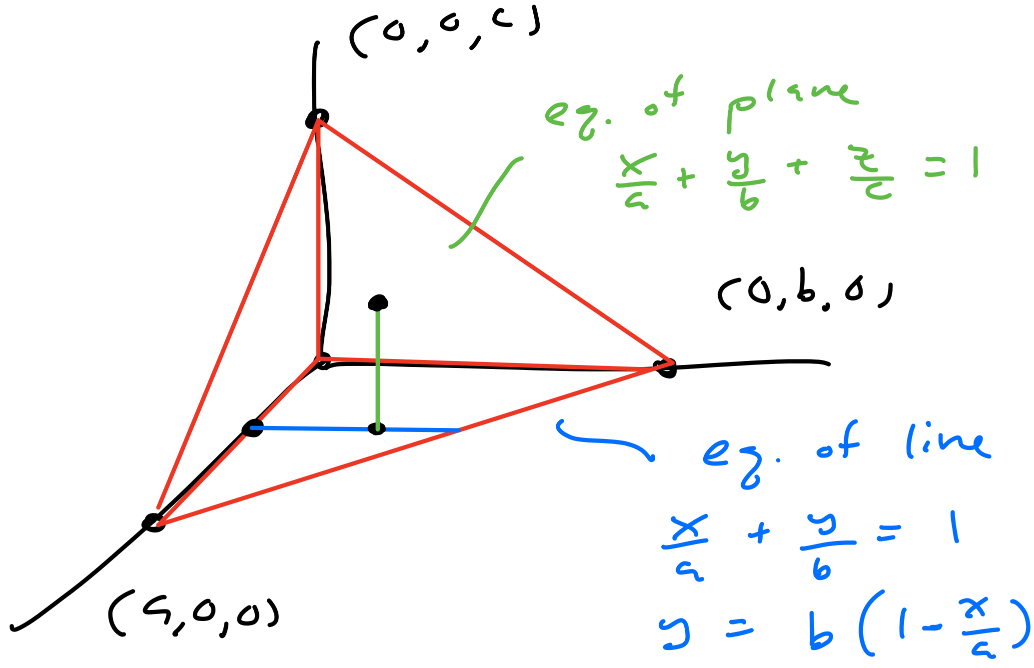
$$= (0, 0, 0.265).$$

Integrating over a Tetrahedron.

Sometimes there is no really good coordinate system.

Consider tetraheron with

vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$:



6 reasonable ways to parametrize this shape.

Fix $\quad 0 \le x \le a$

Then $\quad 0 \le y \le b\left(1 - \frac{x}{a}\right)$

$\quad 0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

For any scalar field we have

$$\iiint_{\text{tetrahedron}} F \, dV$$

$$= \underbrace{\int_0^a \left(\underbrace{\int_0^{b(1-\frac{x}{a})} \left(\underbrace{\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} f\,dz}_{\text{formula in } x,y} \right) dy}_{\text{some formula in } x} \right) dx}_{\text{just a number.}}$$