HW 4 due Friday.

Review integration in 2D.
Given $f: \mathbb{R}^2 \to \mathbb{R}$ & $D \subseteq \mathbb{R}^2$,
define the integral:

$$\iint_D \overbrace{f \underbrace{dA}_{\text{tiny piece of area}}}^{\text{tiny volume, or tiny piece of mass, ...}}$$

To compute:

- Choose a coordinate system.
- Parametrize the domain D in this coordinate system.
- Actually compute the integral.

In Cartesian coordinates:

$$dA = dx\,dy \qquad \text{“}\partial(x,y)\text{”}$$

Other coordinates ("u,v - substitution")

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} x(u,v) \\ y(u,v) \end{Bmatrix}$$

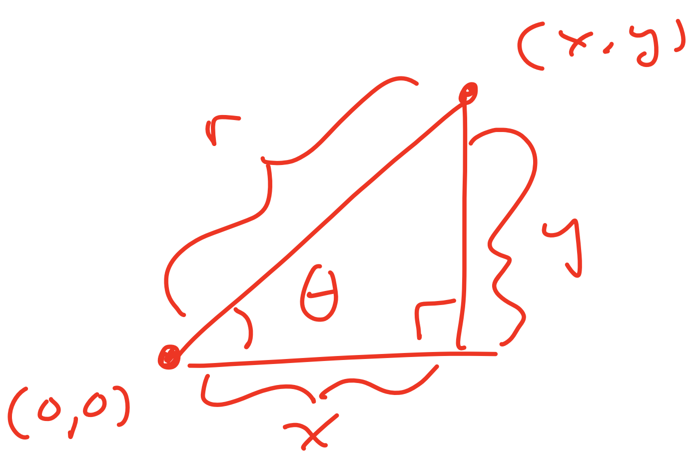
$$dx\,dy = \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right| du\,dv$$

More formally:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

Example: Polar Coords

$$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} r = \sqrt{x^2+y^2} \\ \theta = \arctan(y/x) \end{Bmatrix}$$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$y/x = \sin\theta / \cos\theta$

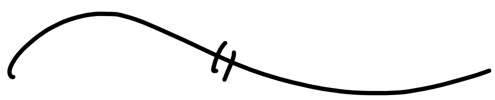
$= \tan\theta$

$$dx\,dy = \left|\det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix}\right| dr\,d\theta$$

just r

$$dr\,d\theta = \left|\det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}\right| dx\,dy$$

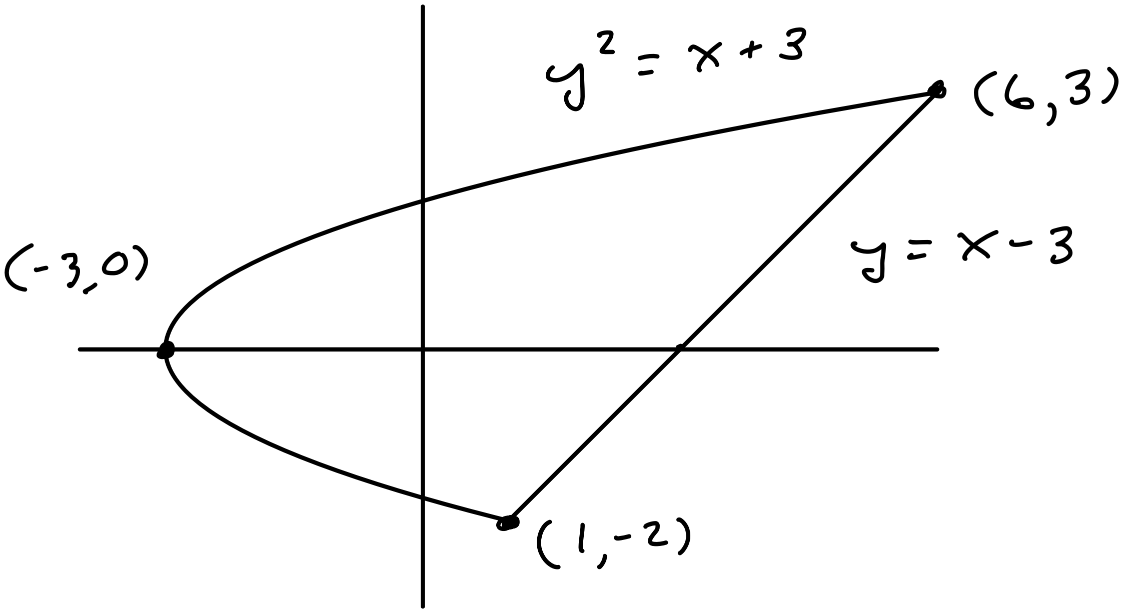
HW 4



Sometimes there is no really good coordinate system. Then we probably use Cartesian coords & brute force.

Example: Integrate $\rho(x,y) = 3x^2 + y^2$ over region between parabola $y^2 = x+3$ and line $y = x-3$

Picture:



Interpretation:

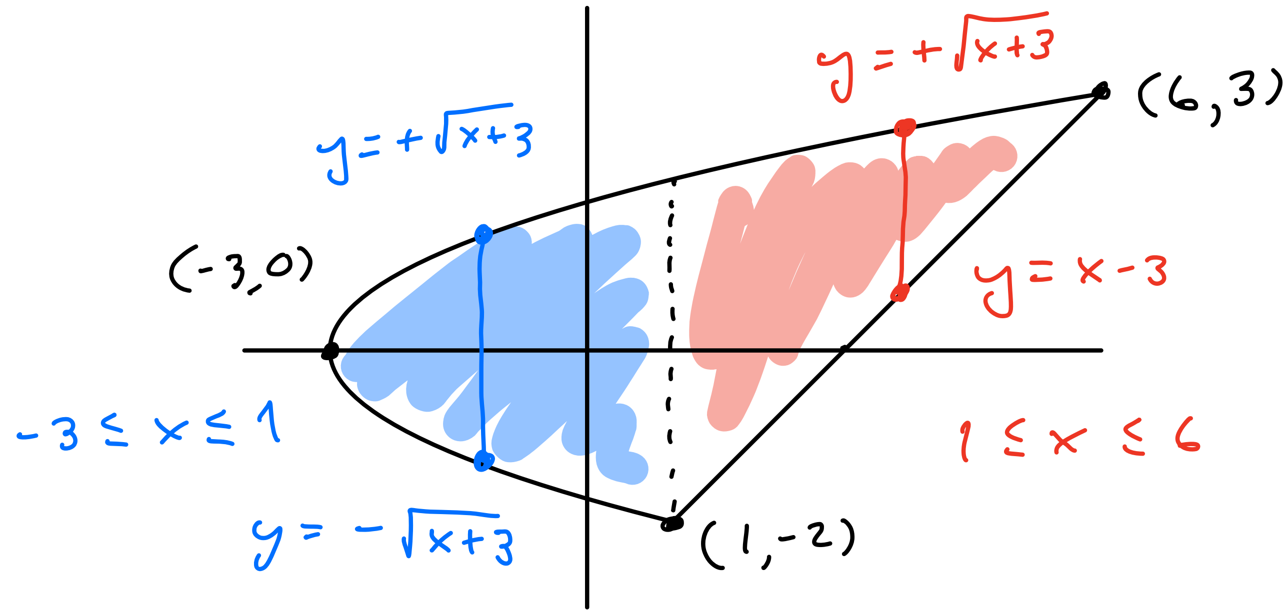
$$\text{mass} = \iint \overbrace{\text{density}\ \underbrace{dA}_{\text{tiny area}}}^{\text{tiny mass}}$$

$$= \iint_D (3x^2 + y^2)\,dx\,dy$$

Parametrize region:

TWO OPTIONS:

- Vertical Slices:



mass = mass of left piece

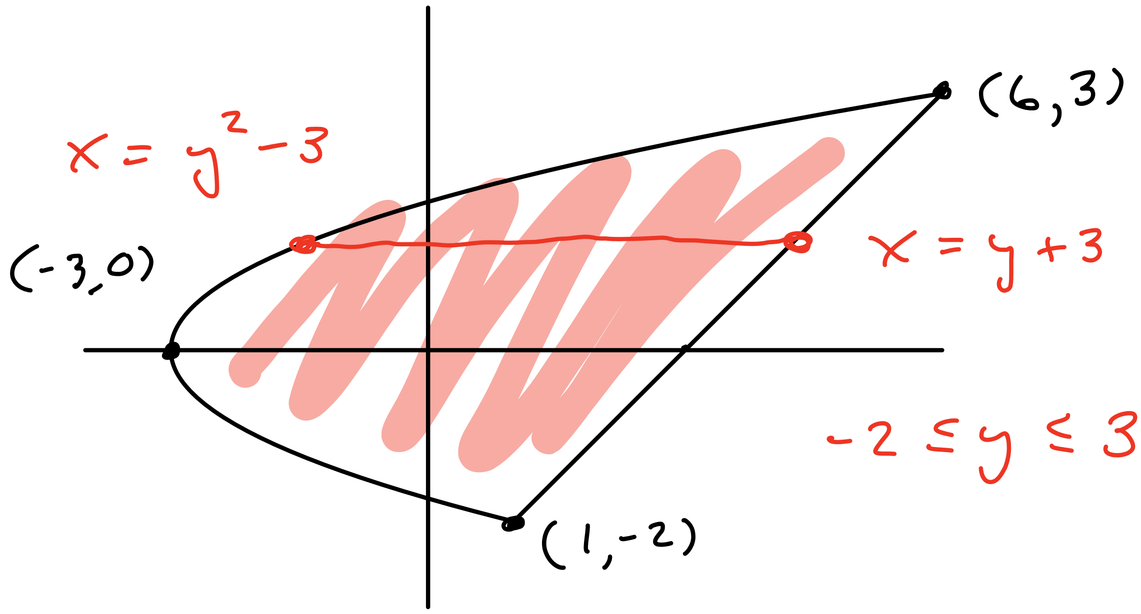
+ mass of right piece

$$= \int_{x=-3}^{1} \left(\int_{y=-\sqrt{x+3}}^{y=+\sqrt{x+3}} (3x^2+y^2)\, dy \right) dx$$

$$+ \int_{x=1}^{6} \left(\int_{y=x-3}^{y=+\sqrt{x+3}} (3x^2+y^2)\, dy \right) dx$$

This looks bad. Skip to next method.

• Horizontal Slices



Two benefits: Only one region
No square roots :)

$$\text{mass} = \int_{y=-2}^{3} \left(\int_{x=y^2-3}^{y+3} (3x^2+y^2)\,dx \right) dy$$

$$= \int_{y=-2}^{3} \underbrace{\left[3\cdot\frac{x^3}{3} + y^2 x \right]_{x=y^2-3}^{x=y+3}}_{\text{expand}} dy.$$

⋮ SKIP
⋮

$$= \int_{-2}^{3} \overbrace{\left(54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6\right)}^{\varphi} dy$$

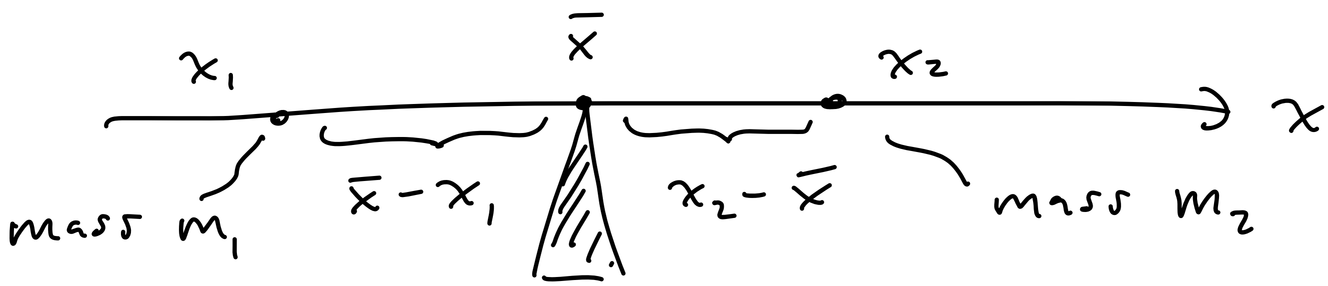
$\vdots$ SKIP (Computer)

$$= \frac{2375}{7} \approx 339 \text{ units of mass.}$$

What is the center of mass ?

[This is the point that follows parabolic trajectory when object is thrown in the air.]

Archimedes :



Law of lever says

balance $\iff$ $m_1(\bar{x}-x_1) = m_2(x_2-\bar{x})$

Solve for $\bar{x}$:

$$m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x}$$

$$(m_1+m_2)\bar{x} = m_1x_1 + m_2x_2$$

$$\bar{x} = \frac{m_1x_1+m_2x_2}{m_1+m_2}$$

Generalize to many point masses:

$$\bar{x} = \frac{m_1x_1+m_2x_2+\cdots+m_nx_n}{m_1+m_2+\cdots+m_n}$$

$$= \frac{\sum m_ix_i}{\sum m_i}$$ total mass

For a continuous density $\rho(x)$ on the real line we get

$$\bar{x} = \frac{\int x \rho(x)\, dx}{\int \rho(x)\, dx}$$ total mass

Given $\rho(x,y)$ in 2D we will use the notation

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

where

$$m = \iint \rho(x,y)\, dx\, dy = \text{total mass}$$

$$M_y = \iint x \rho(x,y)\, dx\, dy$$

= "moment about the y-axis"
(x-coord is distance from y-axis)

$$M_x = \iint y \rho(x,y)\, dx\, dy$$

= "moment about x-axis"

In our example: $\rho(x,y) = 3x^2 + y^2$

Region: $-2 \le y \le 3$

$$y^2 - 3 \le x \le y + 3$$

$$M_y = \int_{-2}^{3} \left(\int_{y^2-3}^{y+3} x(3x^2 + y^2)\,dx \right) dy$$

$$= 39875/42 \quad (\text{computer})$$

Computer also gives

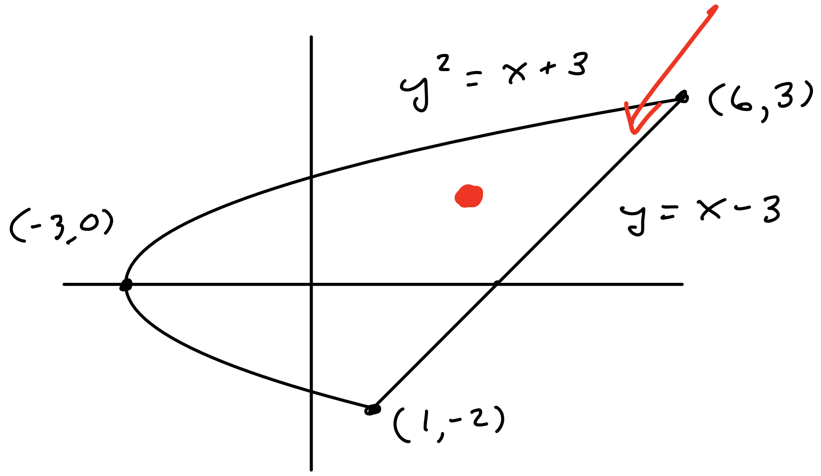
$$M_x = 11125/24$$

So the center of mass is

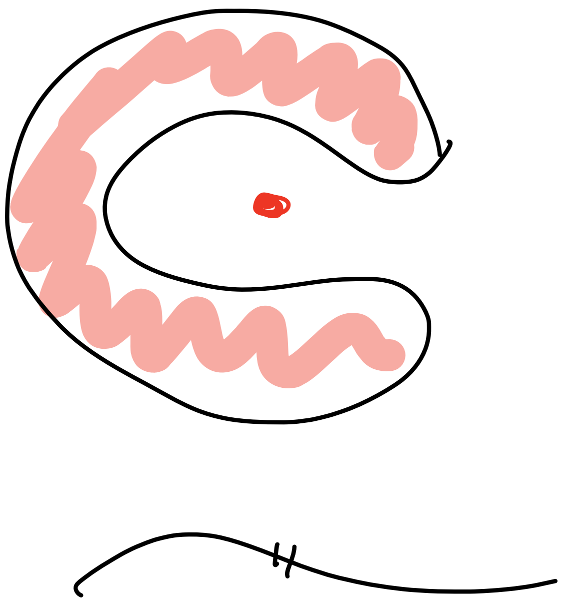
$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

$$= (2.8, 1.4)$$

this tail is heavy



Remark: Center of mass need not be inside the region:



The "same formulas" hold in 3D.

Let $\rho(x,y,z)$ = mass per unit volume.

Then total mass is a triple integral:

$$\underset{M}{\text{mass}} = \iiint \underbrace{\rho(x,y,z)\,\overbrace{dx\,dy\,dz}^{\text{tiny piece of volume}}}_{\text{tiny piece of mass}}$$

The center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

where

$$M_{yz} = \iiint x \rho(x,y,z)\, dx\, dy\, dz$$

= "moment about yz plane"

(x-coord is distance from yz plane)

etc...

HOW to COMPUTE 3D INTEGRAL?

Pretty much the same as 2D.

Example: Volume of a box

$$a_1 \le x \le a_2$$

$$b_1 \le y \le b_2$$

$$c_1 \le z \le c_2$$

$$\text{volume} = \iiint 1\, dx\, dy\, dz$$

$$= \int_{a_1}^{a_2} dx \int_{b_1}^{b_2} dy \int_{c_1}^{c_2} dz$$

$$= (a_2 - a_1)(b_2 - b_1)(z_2 - z_1)$$

(length)(width)(height)

of course.

[Remark: IF the integrand is "separable" $F(x,y,z) = f(x)g(y)h(z)$ then the integral is a product:

$$\iiint F(x,y,z)\,dx\,dy\,dz$$

$$= \iiint f(x)g(y)h(z)\,dx\,dy\,dz$$

$$= \int f(x)\,dx \int g(y)\,dy \int h(z)\,dz.$$

USEFUL!

e.g. $F(x,y,z) = x^2 e^y \sin(z)$

is separable.

$$F(x,y,z) = e^{xy}\sin(yz)$$

is not separable.]

Center of Mass ?

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}, \frac{c_1+c_2}{2}\right)$$

SHOULD BE.

(Check)

$$M_{yz} = \int\int\int x \, dx\, dy\, dz$$

$$= \int x\,dx \int dy \int dz$$

$$= \left(\frac{a_2^2 - a_1^2}{2}\right)(b_2 - b_1)(c_2 - c_1)$$

$$\frac{M_{yz}}{m} = \frac{\left(\frac{a_2^2 - a_1^2}{2}\right)\cancel{(b_2-b_1)}\cancel{(c_2-c_1)}}{(a_2-a_1)\cancel{(b_2-b_1)}\cancel{(c_2-c_1)}}$$

FACTOR

$$= \frac{(\cancel{a_2 - a_1})(a_2 + a_1)}{2} \cdot \frac{1}{\cancel{a_2 - a_1}}$$

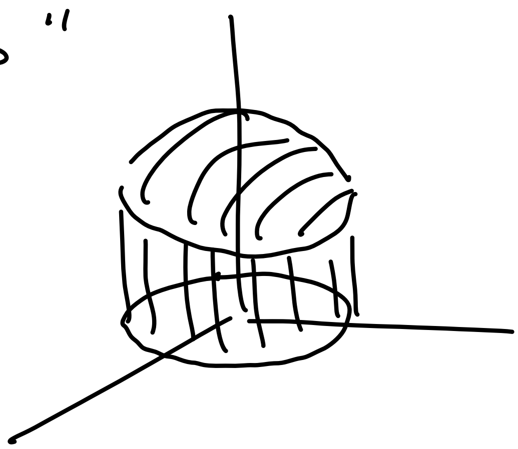
$$= (a_1 + a_2)/2$$

Harder Example:

Compute volume of 3D region E

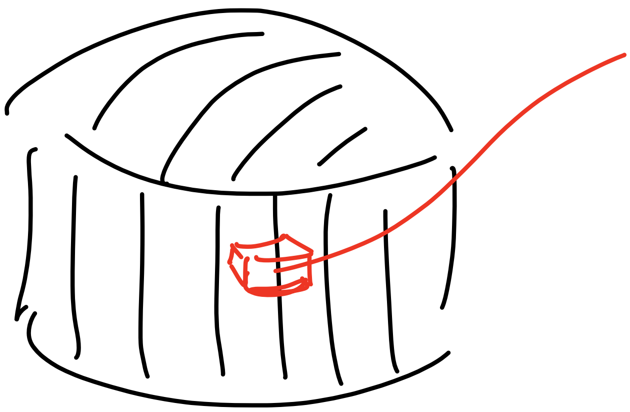
- Above xy plane
- Inside cylinder $x^2 + y^2 \leq 1$
- Inside sphere $x^2 + y^2 + z^2 \leq 4$.

"silo"



We could do this with 2D

integral using polar coords, but we'll use 3D integral for illustration.



tiny piece of volume

$dx\,dy\,dz$

$=$

$r\,dr\,d\theta\,dz$

$$\text{Volume} = \iiint 1 \, dx\,dy\,dz$$

$$= \iiint 1 \; r\,dr\,d\theta\,dz.$$

Parametrize E:

$$x^2+y^2 \le 1 \longrightarrow r^2 \le 1$$

$$\longrightarrow r \le 1.$$

$$x^2+y^2+z^2 \le 4$$

$$z^2 \le 4 - x^2 - y^2$$

nice rotational symmetry

$$z^2 \le 4 - r^2$$

$0 \le z \le \sqrt{4-r^2}$

involves r
so integrate over
z before r.

Also $0 \le \theta \le 2\pi$.

$$\text{volume} = \iiint 1 \, r \, dr \, d\theta \, dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \left(\int_0^{\sqrt{4-r^2}} 1 \, dz \right) dr$$

$$= 2\pi \int_0^1 r \sqrt{4-r^2} \, dr$$

$u = 4 - r^2$

$du = -2 r \, dr$ NICE.

$r \, dr = -\frac{1}{2} du$

$$= 2\pi \int_4^3 -\frac{1}{2} \sqrt{u} \, du$$

$\sqrt{u} = u^{1/2}$

$$= 2\pi \left[-\frac{1}{2} \frac{u^{3/2}}{3/2} \right]_4^3$$

$$= 2\pi \left[-\frac{1}{\cancel{2}} \frac{(3)^{3/2}}{3/\cancel{2}} + \frac{1}{\cancel{2}} \frac{\overset{8}{\cancel{(4)^{3/2}}}}{3/\cancel{2}} \right]$$

NOT SO BAD!

$$= 2\pi \left[\frac{8}{3} - \underset{3^{1/2}}{\cancel{\frac{(3)^{3/2}}{3}}} \right]$$

$$= 2\pi \left[\frac{8}{3} - \sqrt{3} \right]$$