Chapter 5: Integration in 2D & 3D.

Recall: Given a scalar field $f(x,y)$ in $\mathbb{R}^2$ and a 2D region $D \subseteq \mathbb{R}^2$ (e.g. rectangle, circle, ..)

"is a subset of"

Then we can integrate $f(x,y)$ over D:

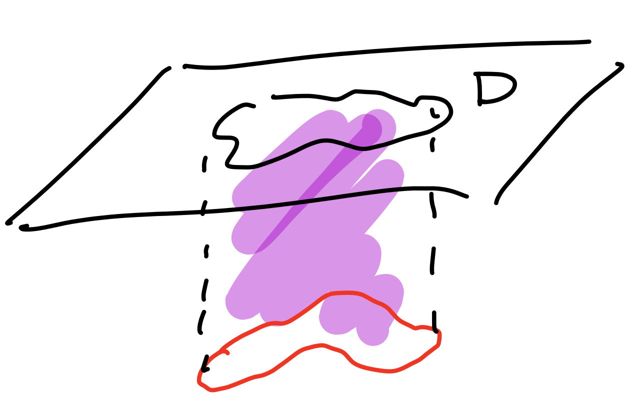
$$I = \iint_D f(x,y)\,dx\,dy = \text{a scalar}$$

Two possible interpretations:

- $f(x,y)$ = height of a surface "above" the x,y plane. Then

  I = "signed volume" of 3D region "above" D and "below" the surface.

I = volume.



$I = -$(volume)

- $f(x,y)$ = mass density
  = mass / unit area.

Then

$$I = \iint_D \underbrace{\overbrace{f(x,y)}^{\text{density}}\,\overbrace{dx\,dy}^{\text{tiny area}}}_{\text{mass of tiny piece}} = \text{total mass of 2D region } D.$$

[Total Mass = $\sum$ point masses

$= \int$ continuous density.]

Can also use this interpretation to compute area. If

density = 1 unit / unit area.

Then

$$\text{area}(D) = \text{total mass}$$

$$= \iint_D 1 \, dx\,dy$$

Integration over rectangles is "easy".

Consider rectangle

$$R = [a_1, a_2] \times [b_1, b_2]$$

= the set of points $(x,y) \in \mathbb{R}^2$ where $a_1 \leq x \leq a_2$ & $b_1 \leq y \leq b_2$.

$$\iint_R f(x,y)\,dxy$$

$$= \int_{y=b_1}^{b_2} \left(\int_{x=a_1}^{a_2} f(x,y)\,dx \right) dy$$

SAME (Fubini's Theorem)

$$= \int_{x=a_1}^{a_2} \left(\int_{y=b_1}^{b_2} f(x,y)\,dy \right) dx$$

[ASIDE: Surface area.

Parametrized surface in 3D.

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$.

Let D be a 2D region in the curved surface. Then

$$\text{area}(D) = \iint_D \underbrace{\|\vec{r}_u \times \vec{r}_v\|\,du\,dv}_{\substack{\text{area of}\\ \text{a tiny piece}\\ \text{of surface}}} .$$

]

Parametrizing a rectangle is easy, but the resulting integral might still be hard.

TRICK: "u-substitution in 2D"

First Example: Polar Coordinates.

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$

(Going back is ugly)

Replace:

$$\underbrace{dx\,dy}_{\text{tiny piece of area}} = r\,\underbrace{dr\,d\theta}_{\text{tiny piece of area}}$$

Sometimes we just write dA for a tiny piece of area. Then we don't have to say what the coordinates are :

$$\iint_{D} F \, dA$$

tiny piece of area.

scalar field in 2D

2D region

Polar Coords work best when region D is a circle, or annulus, of sector of a circle, ...

We used it last time to solve

$$\iint_{x^2+y^2\leq 1} (1-x^2-y^2)\, dx\, dy$$

$$= \iint_{\substack{0\leq r\leq 1\\ 0\leq\theta\leq 2\pi}} (1-r^2)\, r\, dr\, d\theta = \frac{\pi}{2}$$

$0 \leq r \leq 1$
$0 \leq \theta \leq 2\pi$.

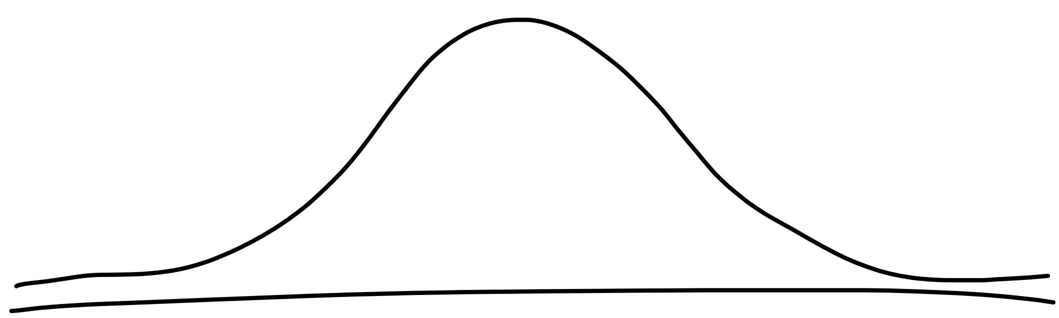
Another Example (Famous Trick):

The indefinite integral $\int e^{-x^2}\,dx$

does not have an elementary formula (i.e. cannot be expressed in terms of polynomials, roots, trig, log, exp). Nevertheless, the definite integral from $-\infty$ to ∞ has a (surprisingly) nice formula:

$$I = \int_{x=-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Very important in statistics:



Normal ("Gaussian") distribution

Function $\frac{1}{\sqrt{\pi}} e^{-x^2}$ has total area 1, so it defines a "random variable".

Integral I is computed with a very clever trick:

$$I^2 = I \cdot I$$

$$= \int_{x=-\infty}^{\infty} e^{-x^2} dx \cdot \int_{y=-\infty}^{\infty} e^{-y^2} dy$$

$$= \iint_{\substack{\text{whole} \\ x,y \text{ plane}}} e^{-x^2} \cdot e^{-y^2} dx\,dy$$

$-r^2$ $\quad\quad$ $r^2 = x^2 + y^2$

$$= \iint e^{-x^2 - y^2} \underbrace{dx\,dy}_{r\,dr\,d\theta}$$

So far this looks silly. But now we change to polar coordinates.

$$= \iint_{\substack{\text{whole plane} \\ 0 \le r \le \infty \\ 0 \le \theta \le 2\pi}} e^{-r^2}\, r\, dr\, d\theta$$

Miracle! This can be integrated using "u-sub".

$$= \int_{\theta=0}^{2\pi} d\theta \cdot \int_{r=0}^{\infty} r e^{-r^2}\, dr$$

[Fact: $\iint f(r)\, g(\theta)\, dr\, d\theta$

$= \int g(\theta)\, d\theta \cdot \int f(r)\, dr.$]

$$= 2\pi \cdot \int_{r=0}^{\infty} r e^{-r^2}\, dr$$

$u = r^2$

$du = 2r\, dr \qquad dr = \frac{1}{2} r\, dr.$

$$= 2\pi \int_{u=0}^{\infty} \frac{1}{2} e^{-u}\, du$$

$$= 2\pi \left[-\frac{1}{2} e^{-u} \right]_{u=0}^{u=\infty}$$

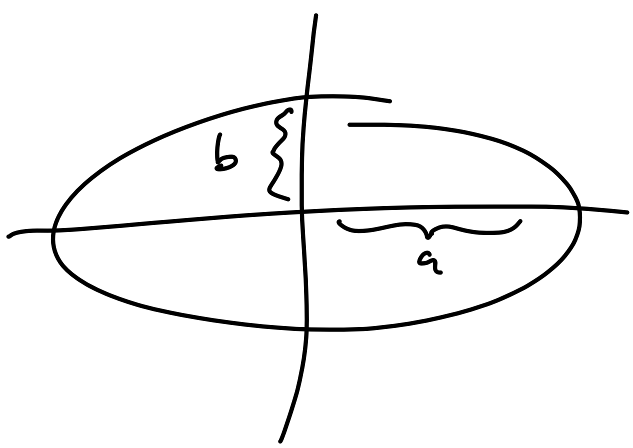
$$= 2\pi \left[-\frac{1}{2} \cancel{e^{-\infty}}^{\,0} + \frac{1}{2} \cancel{e^{0}}^{\,1} \right]$$

$$= 2\pi \left(\frac{1}{2} \right) = \pi \quad ✓$$

NICE!

Try to compute the area of an ellipse.

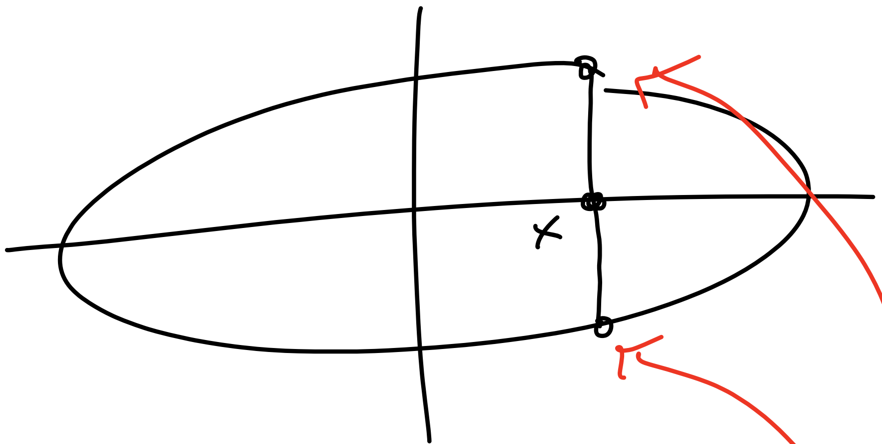
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



Let D be interior of ellipse.

$$\text{area}(D) = \iint_D dx\,dy.$$

How hard could it be?



$$-a \leq x \leq a$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{1 - x^2/a^2}$$

So

$$\text{area}(D) = \int_{x=-a}^{a} \left(\int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{+b\sqrt{1-\frac{x^2}{a^2}}} 1\,dy \right) dx$$

LOOKS BAD!

TRY POLAR COORDS:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$r^2\frac{\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1$$

$$r^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) = 1.$$

Parametrize: $0 \leq \theta \leq 2\pi$

some bad function of θ $\leq r \leq$ some bad function of θ.

Problem: We really want to use

$$\cos^2\theta + \sin^2\theta = 1.$$

Seems to be a really easy idea.

Let $u = x/a$ & $v = y/b$.

Then

$$\text{area} = \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} 1 \, dxy$$

$$= \iint_{u^2+v^2=1} 1 \, (du\,dv)^{?}$$

Question:

$$dx\,dy = \; ? \; du\,dv \, .$$

General Change of Coords in 2D
("u, v substitution")

Let $u(x,y)$ & $x(u,v)$
$v(x,y)$ $\quad$ $y(u,v)$

Chain Rule says

$$dx = \frac{dx}{du} \cdot du + \frac{dx}{dv} \cdot dv$$

$$dx = x_u \cdot du + x_v \cdot dv$$

$$dy = y_u \cdot du + y_v \cdot dv$$

Jacobian Matrix

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

matrix multiplication

ROUGHLY

$$\text{"}dx\,dy\text{"} = \left| \underbrace{\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}}_{\text{area stretch factor.}} \right| \text{"}du\,dv\text{"}$$

Example: Polar Coords

$x = r\cos\theta$

$y = r\sin\theta$

$x_r = \cos\theta$

$y_r = \sin\theta$

$x_\theta = -r\sin\theta$

$y_\theta = r\cos\theta$

$$dx\,dy = \left|\det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix}\right| dr\,d\theta$$

$$= \left|\det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| dr\,d\theta$$

$$= \left(r\cos^2\theta + r\sin^2\theta\right) dr\,d\theta$$

$$= r\left(\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right) dr\,d\theta$$

$$= r\,dr\,d\theta \quad \checkmark$$

This is the "real" way to do it.

Try to go backwards:

$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan(y/x)$$

$$dr\,d\theta = \left|\det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}\right| dx\,dy.$$

It should be $\frac{1}{r}$. You will check on HW4 that it is.

[In general: Matrices

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \;\&\; \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$ <u>are inverses.</u>]

Back to the area of the ellipse

$$D: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Change coordinates

$u = x/a$ $\qquad x = au \qquad x_u = a$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_v = 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad y_u = 0$

$v = y/b$ $\qquad y = bv \qquad y_v = b$

$$dxdy = \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right| dudv$$

[Mnemonic $dx = x_u\, du + x_v\, dv$]

$$\left[\ \left|\det\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|\ \right]$$

don't worry about columns & rows.

$$= \left|\det\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right| du\,dv$$

$$= ab\,du\,dv \,.$$

$$dx\,dy = ab\,du\,dv$$

HOW NICE !

Finally: Area of Ellipse:

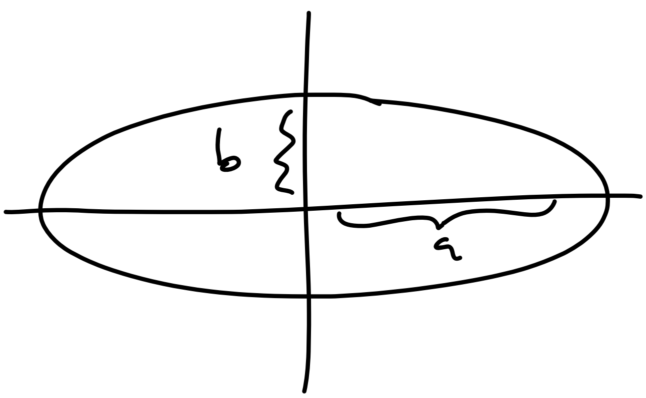
$$\iint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1} 1\,dx\,dy$$

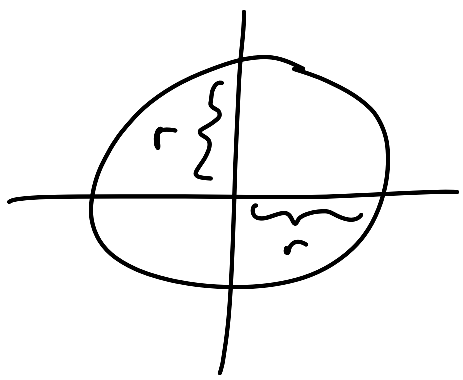
$$= \iint_{u^2+v^2=1} ab\,du\,dv \,.$$

$$= ab \underbrace{\iint_{u^2+v^2=1} 1 \, du \, dv}_{\substack{\text{area of} \\ \text{unit circle,} \\ = \pi}}$$

$$= \pi a b .$$

Compare to area of circle:





area $= \pi a b$ area $= \pi r^2$.

[Remark: Perimeter is much harder. Perimeter of an ellipse is a totally new kind of function.]

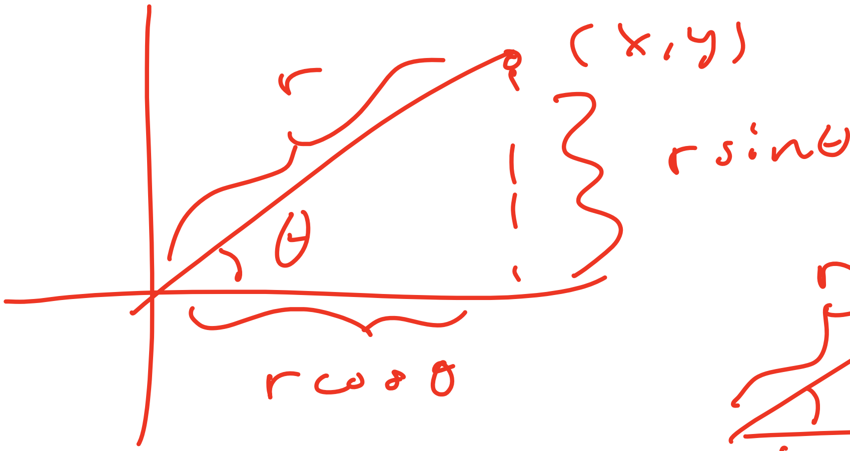
Same ideas work in 3D.

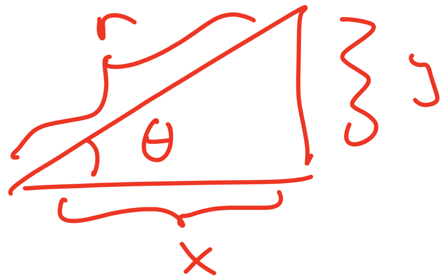
$u(x,y,z)$ $\quad$ $x(u,v,w)$

$v(x,y,z)$ $\quad$ $y(u,v,w)$

$w(x,y,z)$ $\quad$ $z(u,v,w)$

$$\underbrace{dx\,dy\,dz}_{\text{tiny volume}} = \left|\underbrace{\det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}}_{\text{volume stretch factor is determinant of 3x3 Jacobian matrix.}}\right| du\,dv\,dw$$





$x = r\cos\theta$
$y = r\sin\theta$

$r^2 = x^2 + y^2.$