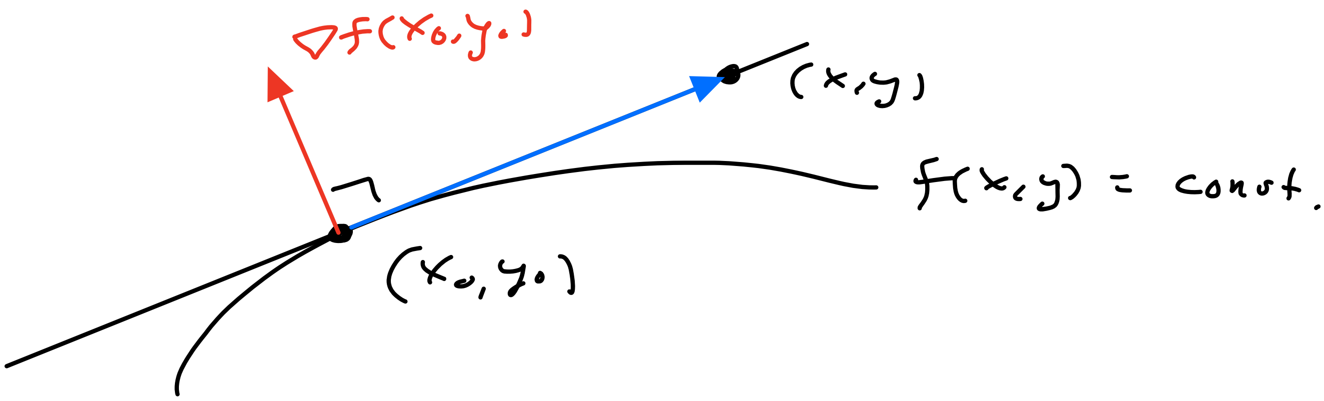
HW 3 Discussion
& Quiz 3 Review.

Problem 1:

Tangent line to curve $f(x,y) = \text{const.}$
at point (x_0, y_0) is

$$\nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) = 0$$

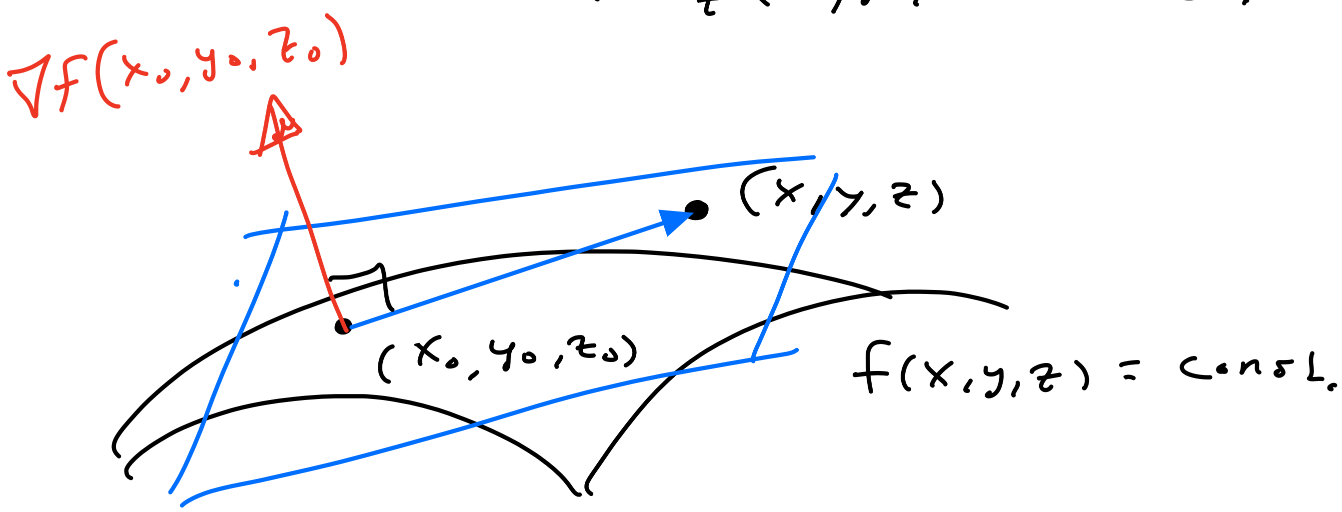


Tangent plane to surface
$F(x,y,z) = \text{const}$ at (x_0,y_0,z_0)

$$\nabla F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$F_x(x_0,y_0,z_0)(x-x_0)$$
$$+ F_y(x_0,y_0,z_0)(y-y_0)$$

$$+ f_z(x_0,y_0,z_0) = 0.$$



In Problem 1 have curve

$$ax^2 + by^2 = 1$$

$$f(x,y) = \text{const.}$$

$$\nabla f(x,y) = \langle 2ax, 2by \rangle.$$

Tangent line at (x_0,y_0) is

$$\nabla f(x_0,y_0) \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$\langle 2ax_0, 2by_0 \rangle \cdot \langle x-x_0, y-y_0 \rangle = 0$$

$$2ax_0(x-x_0) + 2by_0(y-y_0) = 0$$

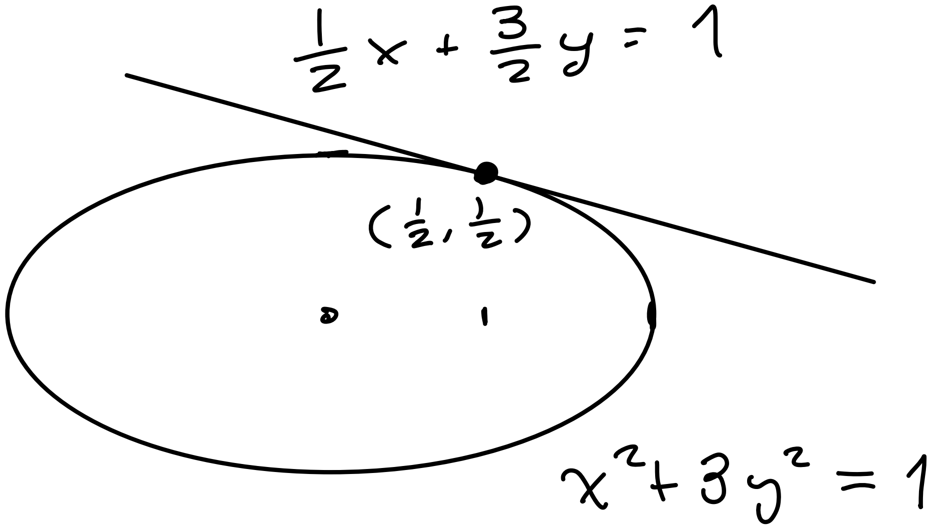
$$2ax_0x + 2by_0y - 2(\cancelto{1}{ax_0^2+by_0^2}) = 0$$

(x_0,y_0) is on the curve!

$$a x_0 x + b y_0 y = 1 \quad \checkmark$$

Special Case : $a = 1$, $b = 3$, $(x_0, y_0) = (\frac{1}{2}, \frac{1}{2})$.

$1/\sqrt{3} = 0.58$



Problem 2 : Chain Rule !

Many equivalent statements.

$f(x,y)$ & $x(t)$, $y(t)$

Implicitly, $f(t)$ function of t.

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt} .$$

$f(x,y,z)$ & $x(t), y(t), z(t)$

$$\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt} + \frac{df}{dz}\cdot\frac{dz}{dt}$$

Complete generality

$f(x_1, \ldots, x_n)$

$x_1(t), x_2(t), \ldots, x_n(t)$

$$\frac{df}{dt} = \frac{df}{dx_1}\cdot\frac{dx_1}{dt} + \cdots + \frac{df}{dx_n}\cdot\frac{dx_n}{dt}$$

More Conceptual:

$f(x_1, \ldots, x_n)$ scalar field.

$\vec{r}(t) = \langle x_1(t), \ldots, x_n(t)\rangle$

is a parametrized path.

Temperature at time t:

$f(\vec{r}(t)) = f(x_1(t), \ldots, x_n(t))$.

Rate of change of temp at time t.

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$\frac{df}{dt} = \left\langle \frac{df}{dx_1}, \dots, \frac{df}{dx_n} \right\rangle \cdot \left\langle x_1'(t), \dots, x_n'(t) \right\rangle$$

$$= \left\langle \frac{df}{dx_1}, \dots, \frac{df}{dx_n} \right\rangle \cdot \left\langle \frac{dx_1}{dt}, \dots, \frac{dx_n}{dt} \right\rangle$$

$$= \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \dots + \frac{df}{dx_n} \cdot \frac{dx_n}{dt}.$$

SAME!

Problem 2: Practice with not thinking.

$$\frac{df}{dr} = \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr}$$

(Ignore θ!)

$$f_r = f_x \cdot \frac{dx}{dr} + f_y \cdot \frac{dy}{dr}$$

$$x(r,\theta) = r\cos\theta \longrightarrow dx/dr = \cos\theta$$

$$y(r,\theta) = r\sin\theta \longrightarrow dy/dr = \sin\theta$$

$$f_r = f_x \cdot \cos\theta + f_y \cdot \sin\theta.$$

$f_r(r,\theta)$

$f_x(x,y)$ or $f_x(r,\theta)$

$f_y(x,y)$ or $f_y(r,\theta)$

$\Big\}$ don't have formulas for these.

$$F_{rr} = (f_x \cdot \underset{\text{const.}}{\cos\theta} + f_y \cdot \underset{\text{const.}}{\sin\theta})_r$$

$$= f_{xr} \cdot \cos\theta + f_{yr} \cdot \sin\theta .$$

? ?

Want in terms of f_{xx}, f_{yy}, f_{xy}.

Think of $f_x(x,y)$ as function of x & y, use chain rule

$$\frac{df_x}{dr} = \frac{df_x}{dx} \cdot \frac{dx}{dr} + \frac{df_x}{dy} \cdot \frac{dy}{dr}$$

$$f_{xr} = f_{xx} \cdot \frac{dx}{dr} + f_{xy} \cdot \frac{dy}{dr}$$

$$= f_{xx} \cdot \cos\theta + f_{xy} \cdot \sin\theta$$

Similarly,

$$f_{yr} = f_{yx} \cdot \frac{dx}{dr} + f_{yy} \cdot \frac{dy}{dr}$$

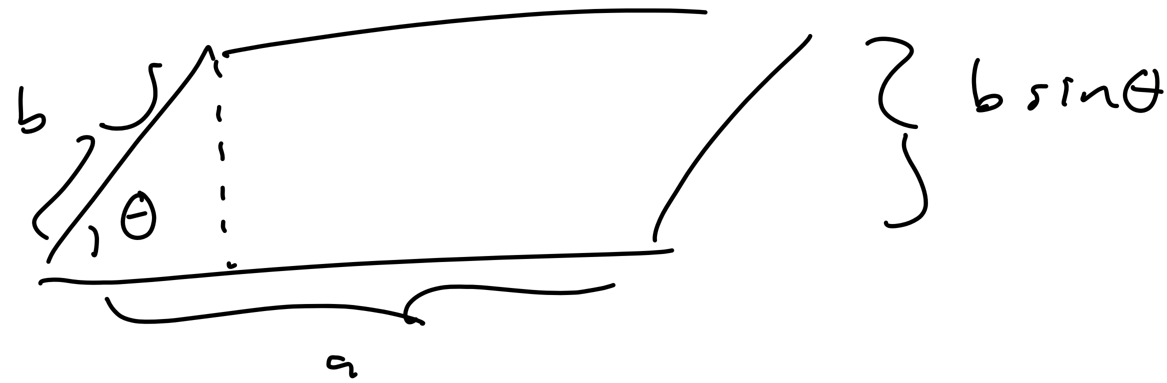
$$= f_{yx} \cdot \cos\theta + f_{yy} \cdot \sin\theta .$$

Finally:

$$F_{rr} = f_{xx} \cdot \cos^2\theta + 2f_{xy} \cdot \sin\theta\cos\theta + f_{yy} \cdot \sin^2\theta .$$

Linear Approximation:

$A(a, b, \theta) = ab \sin\theta = \text{base} \cdot \text{height}$



Surgery to get a rectangle:

Tiny changes da, db, $d\theta$.
related to tiny change dA
via the chain rule

$$dA = \frac{dA}{da} \cdot da + \frac{dA}{db} \cdot db + \frac{dA}{d\theta} \cdot d\theta$$

$$= b\sin\theta\, da + a\sin\theta\, db + ab\cos\theta\, d\theta.$$

Application:

$$a = 2 \pm 0.1 \text{ cm} \quad (da = 0.1)$$

$$b = 1 \pm 0.1 \text{ cm} \quad (db = 0.1)$$

$$\theta = 45 \pm 1 \text{ degrees} \quad (d\theta = 1^\circ)$$

In radians:

$$d\theta = 1^\circ = 1/360 \text{ of } 2\pi$$

$$= 2\pi/360 = 0.0172\ldots$$

Plugging everything in:

$$dA = 2\cdot \overset{\sqrt{2}/2}{\cancel{\sin(45^\circ)}}(0.1)$$

$$+ 1\cdot \overset{\sqrt{2}/2}{\cancel{\sin(45^\circ)}}(0.1)$$

$$+ (2)(1)\overset{\sqrt{2}/2}{\cancel{\cos(45^\circ)}}(0.0172)$$

$$= \sqrt{2}(0.1) + \frac{\sqrt{2}}{2}(0.1)$$

$$+ \sqrt{2}(0.0172)$$

$$= 0.23648\ldots$$

$$A = ab\sin\theta = 2\frac{\sqrt{2}}{2} = \sqrt{2} = 1.41$$

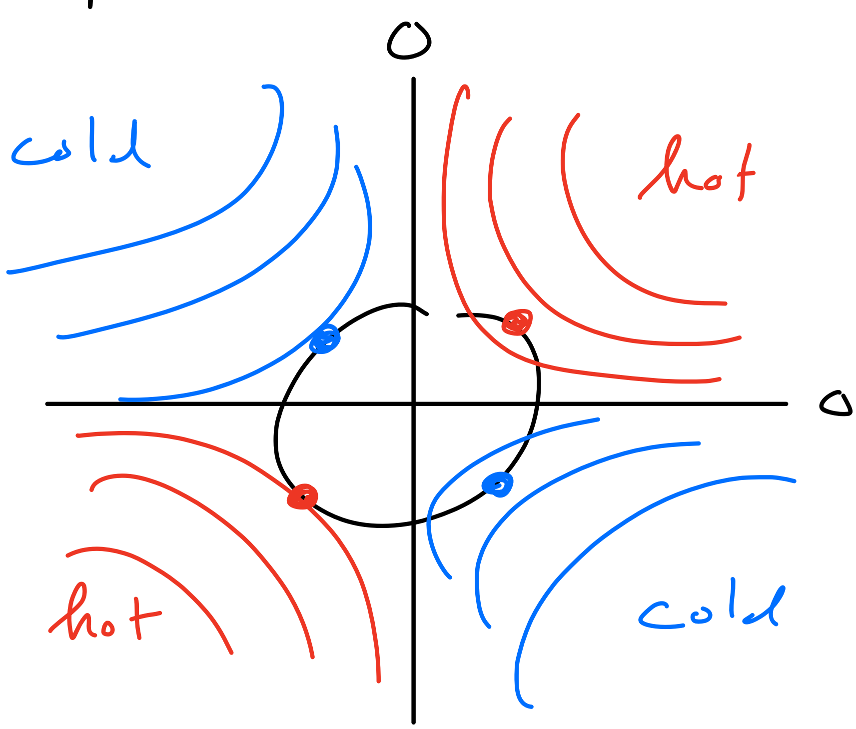
The area is

$$1.41 \pm 0.24 \text{ cm}^2$$

Problem 4: Constrained Optimization.

Maximize / Minimize $f(x,y) = xy$ subject to constraint $x^2 + y^2 = 1$.

Picture: f = temp. Want hottest / coldest points on unit circle:



First method: Parametrize the path

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$f(\vec{r}(t)) = \cos t \sin t = T(t)$$

$$T(t) = \frac{1}{2} \sin(2t)$$

$$T'(t) = \frac{1}{2} \cos(2t) \cdot 2$$

$$= \cos(2t)$$

Calc I: To optimize $T(t)$, set

$$T'(t) = 0$$

$$\cos(2t) = 0$$

$$2t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$t = \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

[Remark: Given angle θ there are n different angles that deserve to be called θ/n.]

Second deriv test:

$$T''(t) = -2\sin(2t)$$

$$T''\left(\frac{\pi}{4}\right) < 0$$

$$T''\left(\frac{3\pi}{4}\right) > 0$$

$$T''\left(\frac{5\pi}{4}\right) < 0$$

$$T''\left(\frac{7\pi}{4}\right) > 0$$

Local max at

$$\vec{r}\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{r}\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Local min at

$$\vec{r}\left(\frac{3\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\vec{r}\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Calc III Method:

Constraint $x^2 + y^2 = 1$

$$g(x,y) = \text{const.}$$

Lagrange Multipliers:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\langle y, x \rangle = \lambda \langle 2x, 2y \rangle$$

$$\begin{cases} y = \lambda 2x \\ x = \lambda 2y \\ x^2 + y^2 = 1 \end{cases}$$

$$\lambda = \frac{y}{2x} = \frac{x}{2y}$$

$$2y^2 = 2x^2$$

$$y^2 = x^2.$$

Then $x^2 + y^2 = 1$

$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x^2 = 1/2$$

$$x = \pm 1/\sqrt{2} \quad \checkmark$$

Then y comes from $y^2 + x^2 = 1$.

But using this method it's harder to test whether max or min.

Problem 5: Unconstrained Optimization.

Optimize $f(x,y)$. Full stop.

$f(x,y) = x^3 + 2xy - 4y^2 - 6x$

$f_x = 3x^2 + 2y - 6$

$f_y = 2x - 8y$

$f_{xx} = 6x$

$f_{yy} = -8$

$f_{xy} = 2$ of course!

$f_{yx} = 2$

Critical Points

$\nabla f(x,y) = \langle 0, 0 \rangle$

$\langle 3x^2 + 2y - 6,\ 2x - 8y \rangle = \langle 0, 0 \rangle$

$2x - 8y = 0 \rightsquigarrow y = x/4.$

$3x^2 + 2y - 6 = 0$

$3x^2 + 2(x/4) - 6 = 0$

$$3x^2 + x/2 - 6 = 0$$

Miracle!

$$6x^2 + x - 12 = 0$$

$$x = \frac{-1 \pm \underbrace{\sqrt{1^2 - 4\cdot 6(-12)}}_{=17}}{12}$$

$$x = \frac{-1-17}{12} = \frac{-18}{12} = \frac{-3}{2}$$

$$\text{or } \frac{-1+17}{12} = \frac{16}{12} = \frac{4}{3}$$

TWO CRITICAL POINTS

$$\left(\frac{-3}{2}, \frac{-3}{8}\right) \quad \& \quad \left(\frac{4}{3}, \frac{1}{3}\right).$$

Second Derivative Test:

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} 6x & 2 \\ 2 & -8 \end{pmatrix}$$

$$\det(Hf) = -48x - 4$$

$\det(Hf)\left(-\frac{3}{2}, -\frac{3}{8}\right) = -48\left(-\frac{3}{2}\right) - 4 > 0$

Max or min.

$f_{xx}\left(-\frac{3}{2}, -\frac{3}{8}\right) = 6\left(-\frac{3}{2}\right) < 0$

MAX.

$f_{yy}\left(-\frac{3}{2}, -\frac{3}{8}\right) = -8 < 0$

only need to check one.

$\det(Hf)\left(\frac{4}{3}, \frac{1}{3}\right) = -48\left(\frac{4}{8}\right) - 4 < 0$

SADDLE.