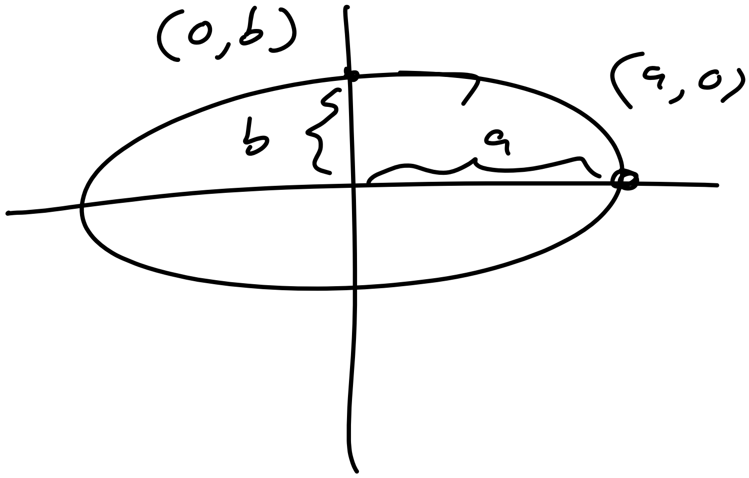
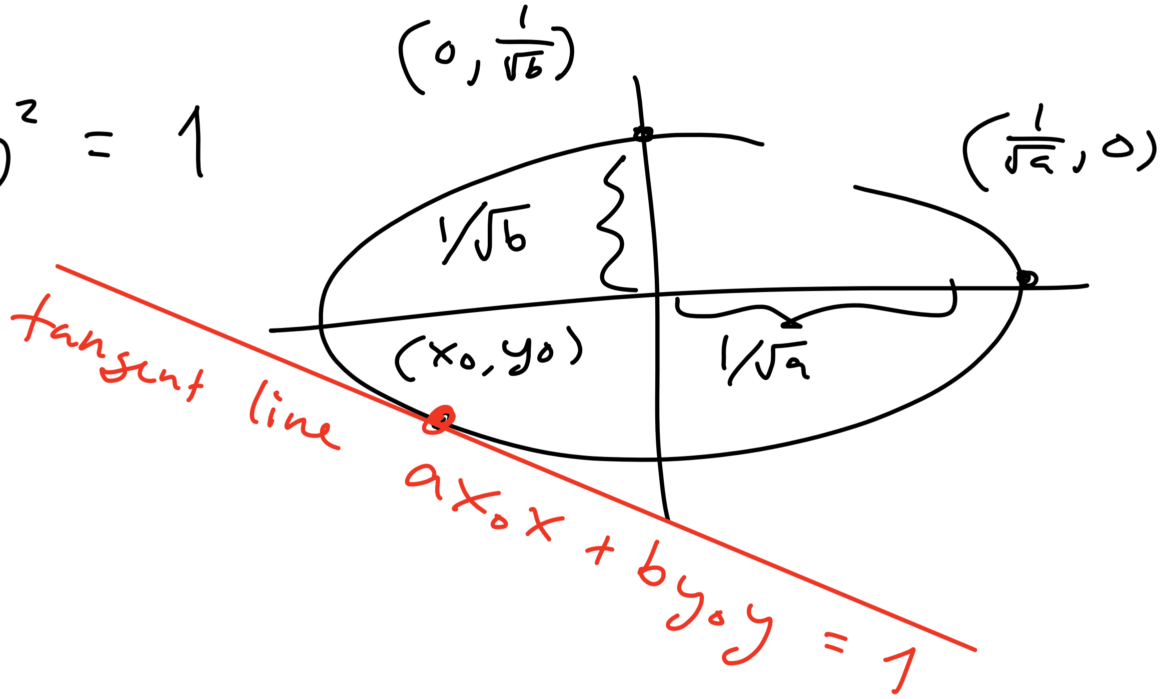
HW 3 Problem 1:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



$$ax^2 + by^2 = 1$$



HW 3 due Friday.

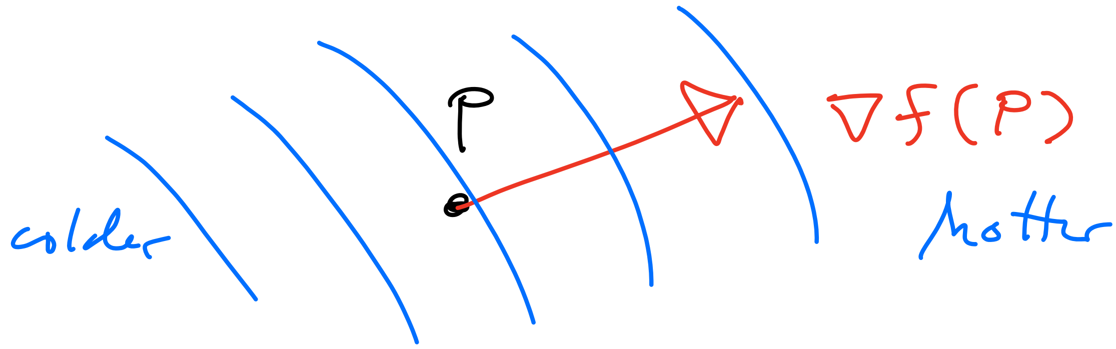
Recall: A scalar field is a function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Assigns scalar $f(P)$ to each point P.

Derivative ∇f assigns a <u>vector</u>

$\nabla f(P)$ to each point P



$f(P)$ = temperature at P
$\nabla f(P)$ = direction of greatest increase of temperature.

Definition:

$$f(x_1, \ldots, x_n)$$

$$\nabla f = \left\langle \frac{df}{dx_1}, \frac{df}{dx_2}, \ldots, \frac{df}{dx_n} \right\rangle.$$

Gradient is $\perp$ to the level curves (curves of constant temperature).

Why?

Multivariable Chain Rule:

$f(P)$ is temperature at point P.

You travel path $\vec{r}(t)$.

Your temperature at time t is

$$T(t) = f(\vec{r}(t)).$$

Your rate of change of temperature at time t is

$$\underbrace{T'(t)}_{\text{scalar}} = \underbrace{\nabla f(\vec{r}(t))}_{\text{vector}} \bullet \underbrace{\vec{r}\,'(t)}_{\text{vector}}$$

dot product.

OR

$$(f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t)$$

composition

$$= (\nabla f \circ \vec{r})(t) \bullet \vec{r}\,'(t)$$

composition of functions

dot product.

$$(f \circ \vec{r})' = (\nabla f \circ \vec{r}) \bullet \vec{r}\,'$$

Consequence: Suppose you travel on a level curve / level surface:

$$f(\vec{r}(t)) = \text{constant}.$$

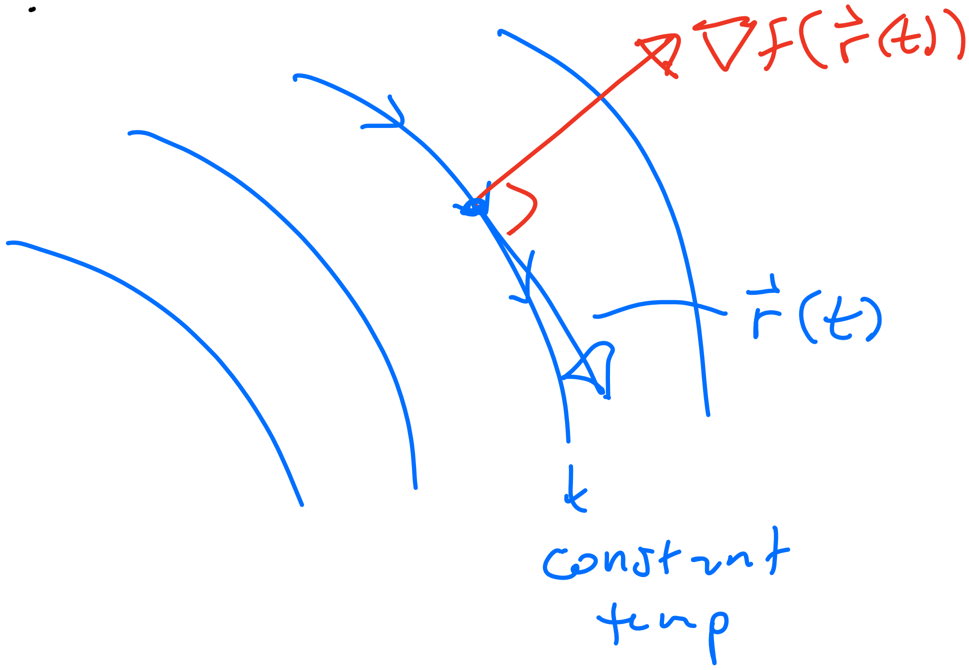
Then

$$\frac{d}{dt}\left[f(\vec{r}(t)) \right] = 0$$

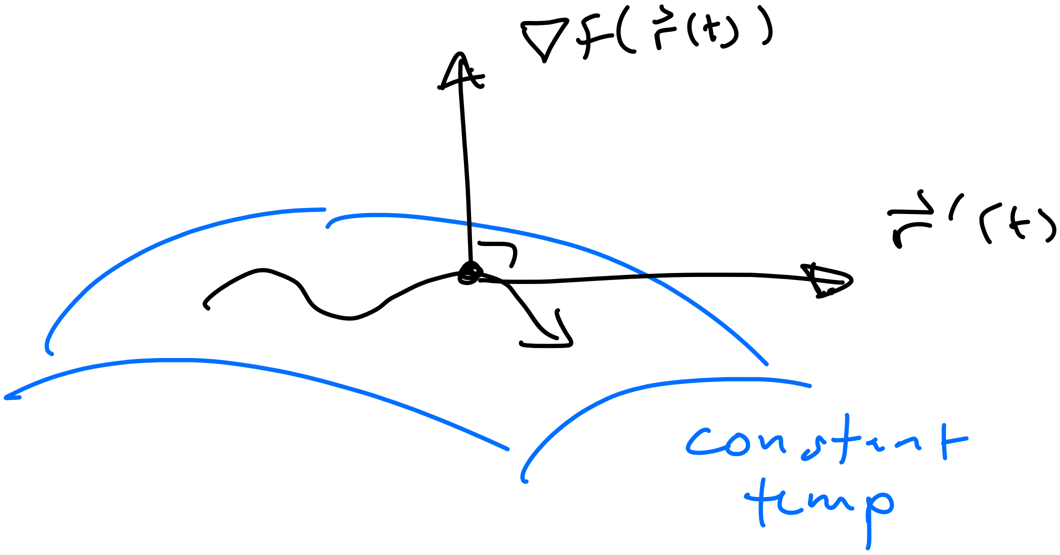
$$\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$$

perpendicular vectors.

Picture:



Picture in 3D: level surfaces



So $\nabla f(P)$ is $\perp$ to the level surface through any point P.

Example: Consider scalar field

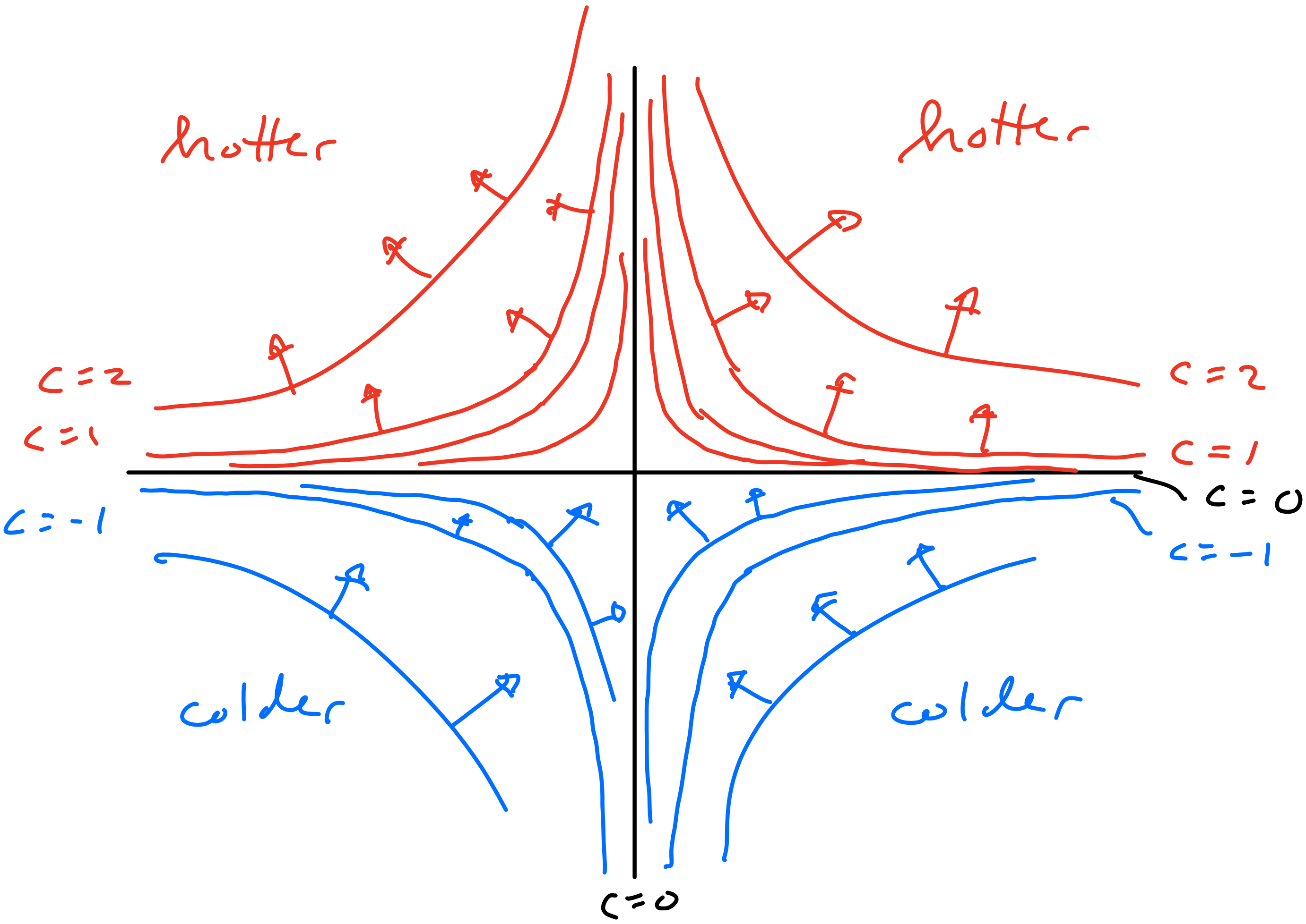
$$f(x,y) = x^2 y.$$

Level curves $f(x,y) = c$ (constant)

$$x^2 y = c$$

$$y = \frac{c}{x^2}$$

What do these curves look like?



Suppose we travel along the curve

$$\vec{r}(t) = \langle t, 2-t^2 \rangle.$$

Our temperature at time t is

$$\begin{aligned} T(t) &= f(\vec{r}(t)) \\ &= f(t, 2-t^2) \\ &= (t)^2(2-t^2) \\ &= 2t^2 - t^4 \end{aligned}$$

When is our temperature maximized or minimized?

$$T'(t) = 0$$

$$4t - 4t^3 = 0$$

$$4t(1-t^2) = 0$$

$$\Rightarrow \quad t = 0 \quad \text{or} \quad 1 - t^2 = 0$$

$$t = \pm 1.$$

Second Derivative:

$$T''(t) = 4 - 12t^2$$

So min at $t = 0$ $\quad [T''(0) > 0]$

max at $t = \pm 1$ $\quad [T''(\pm 1) < 0]$

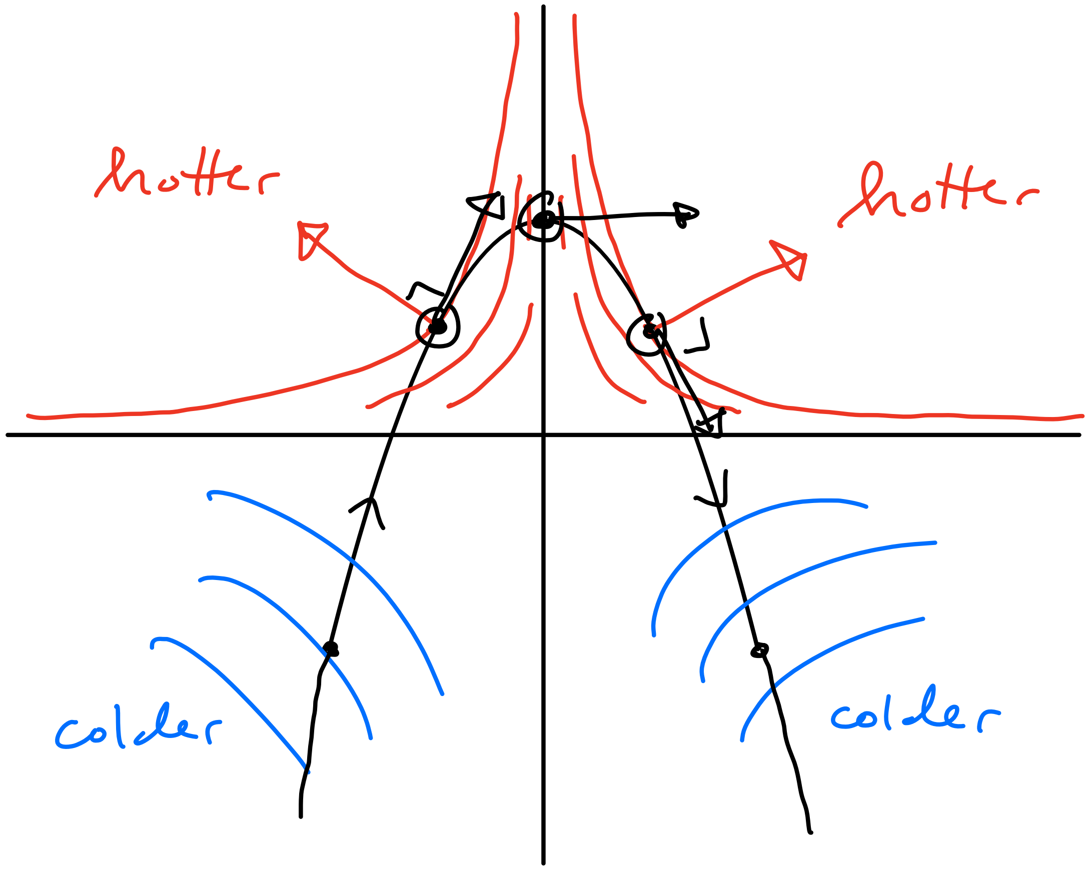
Where is temperature max or min?

$$\vec{r}(0) = \langle 0, 2 \rangle$$

$$\vec{r}(+1) = \langle +1, 2 - (+1)^2 \rangle = \langle 1, 1 \rangle$$

$$\vec{r}(-1) = \langle -1, 2 - (-1)^2 \rangle = \langle -1, 1 \rangle.$$

Picture:



Local maxima happened when our velocity is $\perp$ to gradient.

Indeed, local max $\Longrightarrow$ $T'(t) = 0$

$$T'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$0 = \underbrace{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{perpendicular!}}$$

$t=0$ a bit different because $\nabla f(\vec{r}(0)) = \langle 0, 0 \rangle$.

Every vector is $\perp$ to $\langle 0,0 \rangle$.

So that case is "degenerate".

Another point of view.

Instead of $\vec{r}(t) = \langle t, 2-t^2 \rangle$,

eliminate t. The parabola is

$$y = 2 - x^2$$

$$x^2 + y = 2$$

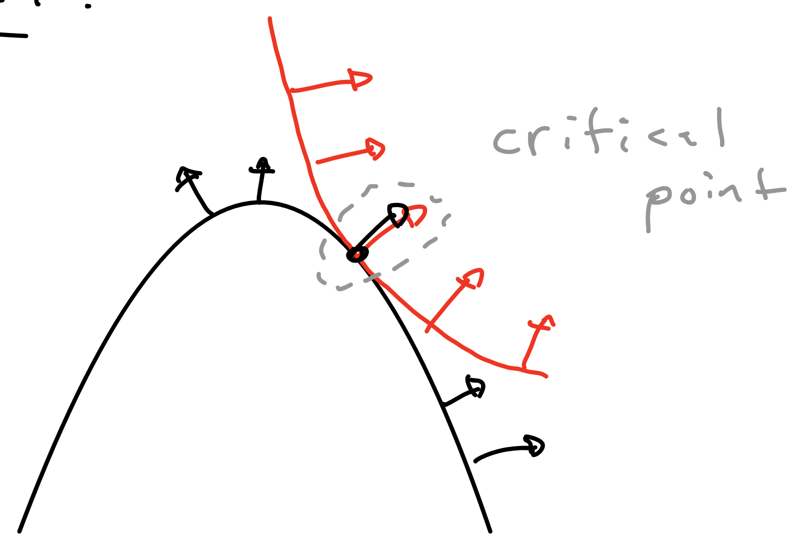
$$g(x,y) = 2$$

for $g(x,y) = x^2 + y$.

In this language, look for

points where $\nabla f(x,y)$ & $\nabla g(x,y)$

are <u>parallel</u>.



This is the method of "Lagrange Multipliers". Calculate:

If $\nabla f(x,y)$ & $\nabla g(x,y)$ are parallel then

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

for some scalar λ.

In our case,

$$f(x,y) = x^2 y \quad \text{(temp.)}$$

$$g(x,y) = x^2 + y \quad \text{(defines our parabola)}$$

$$\nabla f(x,y) = \langle 2xy, x^2 \rangle$$

$$\nabla g(x,y) = \langle 2x, 1 \rangle$$

Set $\langle 2xy, x^2 \rangle = \lambda \langle 2x, 1 \rangle$

$$\langle 2xy, x^2 \rangle = \langle 2x\lambda, \lambda \rangle$$

$$\begin{cases} 2xy = 2x\lambda \\ x^2 = \lambda \end{cases}$$

And we are only interested in points on the parabola $y = 2 - x^2$.

So get 3 equations in 3 unknowns:

$$\begin{array}{l} ① \\ ② \\ ③ \end{array} \begin{cases} 2xy = 2x\lambda, \\ x^2 = \lambda, \\ y = 2 - x^2. \end{cases}$$

In general, VERY HARD to solve.

But this "textbook problem" is not bad.

If $x = 0$ then $y = 2$.

And $(x, y) = (0, 2)$ is a solution.

If $x \neq 0$ then

$$① : \quad \cancel{2x}\, y = \cancel{2x}\, \lambda$$

$$y = \lambda$$

$$② : \quad x^2 = \lambda$$

$$x^2 = y$$

$$③ : \quad y = 2 - x^2.$$

(2) & (3) : $x^2 = 2 - x^2$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1.$$

So two more critical points

$$(x,y) = (+1,+1) \text{ or } (-1,+1).$$

Same solution as before :)

Three critical points

$$(0,2), \quad (1,1), \quad (-1,1)$$

min of f	max of f	max of f

Lagrange Multipliers in General:

Maximize $f(x_1, x_2, \dots, x_n)$

subject to constraint

$$g(x_1, x_2, \dots, x_n) = K$$

Solution: Find all critical

points $(x_1, \dots, x_n)$ such that

$$\begin{cases} \nabla f(x_1, \dots, x_n) = \lambda \nabla g(x_1, \dots, x_n) \\ g(x_1, \dots, x_n) = K \end{cases}$$

In general "impossible" to solve exactly, so use a computer to get numerical solutions.

Linear Approximation:
Another point of view on the chain rule.

$$\frac{d}{dt}\left[f(\vec{r}(t)) \right] = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t).$$

Let's write

$$f(x, y, z)$$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle.$$

$$\vec{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

Then the chain rule says:

$$\frac{dF}{dt} = \nabla f \cdot \vec{r}\,'$$

$$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle.$$

$$\boxed{\frac{dF}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt} + \frac{df}{dz}\cdot\frac{dz}{dt}}$$

PURE ALGEBRA !

2D Version:

$f(x,y)$ function of x & y.

$x(t)$, $y(t)$ Functions of t.

$$\frac{dF}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}.$$

Intuition: $\frac{df}{\cancel{dx}}\cdot\frac{\cancel{dx}}{dt} \approx \frac{df}{dt}$

NOT correct math!

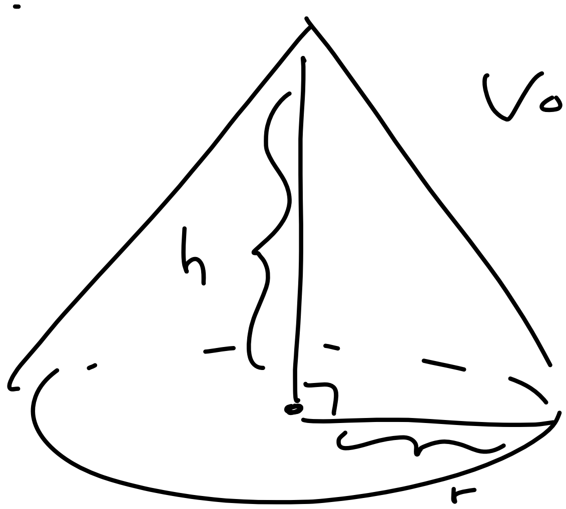
Application:

Consider a right circular cone with height h & radius r.

Volume is a function of h & r:

$$V(r,h) = \frac{1}{3}\pi r^2 h$$

Picture:



$\text{Vol} = \frac{1}{3}\pi r^2 h.$

Suppose h & r change with time: $h(t)$, $r(t)$.

Then the volume changes with time:

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} + \frac{dV}{dh}\cdot\frac{dh}{dt}$$

To simplify terminology, sometimes

we write $V_t = dV/dt$

$V_r = dV/dr$

$V_h = dV/dh.$

$$V_t = V_r \cdot \frac{dr}{dt} + V_h \cdot \frac{dh}{dt}$$

Have $V_r = \frac{1}{3}\pi 2rh$

$V_h = \frac{1}{3}\pi r^2$

So

$$\frac{dV}{dt} = \frac{1}{3}\pi 2rh \cdot \frac{dr}{dt} + \frac{1}{3}\pi r^2 \cdot \frac{dh}{dt}$$

But maybe it's not changing with time; it's changing for some other reason. So let's just say

$$dV = \frac{1}{3}\pi 2rh \cdot dr + \frac{1}{3}\pi r^2 \cdot dh$$

tiny change in V

related to tiny changes in r & h.

Application: Error estimation.

Measure the radius & height:

$$r = 120 \pm 1.8 \text{ in}$$
$$h = 140 \pm 2.5 \text{ in}$$

Then $V = \frac{1}{3}\pi r^2 h \pm dV$

$$V = 2{,}111{,}150 \pm dV$$

how big is the "error"?

Errors are related by chain rule:

$$dV = \frac{dV}{dr} \cdot dr + \frac{dV}{dh} \cdot dh$$

$$dV = \frac{1}{3}\pi 2rh \cdot dr + \frac{1}{3}\pi r^2 \cdot dh.$$

$$dV = \frac{1}{3}\pi 2 (120)(140) \cdot (1.8)$$
$$+ \frac{1}{3}\pi (120)^2 \cdot (2.5)$$

$$= 101{,}033 \text{ in}^3.$$

We conclude that

$$V = 2{,}111{,}150 \pm 101{,}033 \text{ in}^3$$

$$= 2.11 \pm 0.1 \text{ million in}^3.$$