HW 5 due Tues

Quiz 5 on Wed

Final Project due next Fri June 24.

Now: Chapter 6 (Vector Calculus)

Recall: Given vector field

$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$$

and a curve $\vec{r} : \mathbb{R} \to \mathbb{R}^n$, we define the "line integral"

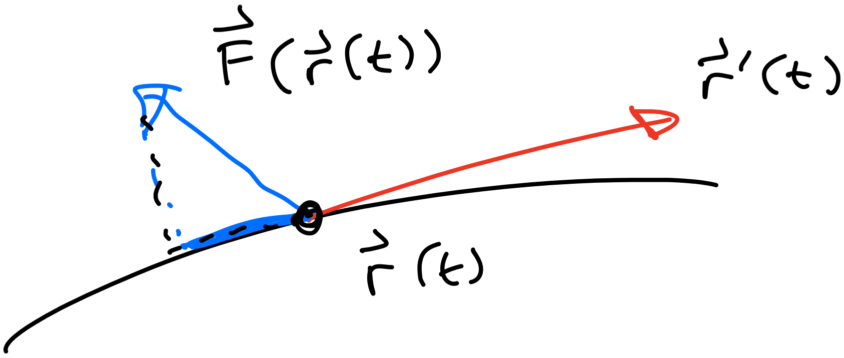
$$\int_{\text{curve}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

= sum the component of $\vec{F}$ along the curve

= "on average, how much does $\vec{F}$ point in the direction of the curve?"

$= 0$ if $\vec{F} \perp$ curve at every point

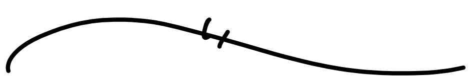
< 0 if $\vec{F}$ points against the curve.



here $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) < 0$

Physics: $\vec{F}$ force field.

$$\int_{\text{curve}} \vec{F} = $$ amount of KE added to particle by the field. ("speed")



Fund Thm Line Integrals:

If $\vec{F} = \nabla f$ then

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start point})$$

Proof:

$$\int_{\text{curve}} \vec{F} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

CHAIN RULE

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt$$

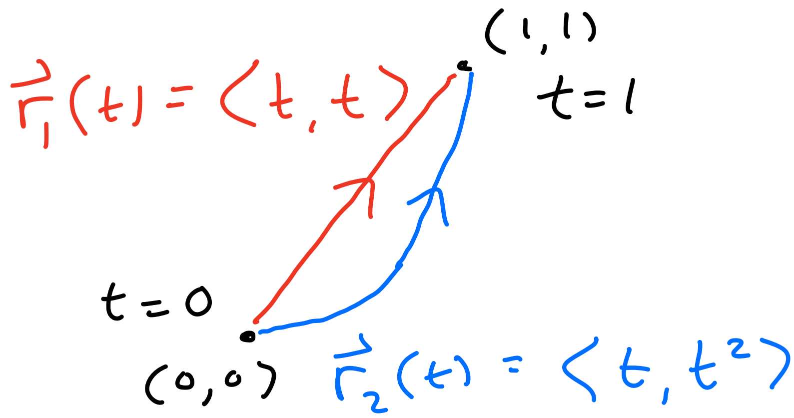
Calc I

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \checkmark$$

Consequence: If $\vec{F} = \nabla f$ then

$\int_{\text{curve}} \vec{F}$ only depends on the endpoints, <u>not</u> on the shape of the curve.

Example:



$$\vec{F} = \nabla(xy + y)$$

$$= \langle y, x+1 \rangle$$

$$\int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1(t)\, dt$$

$$= \int \langle t, t+1 \rangle \cdot \langle 1, 1 \rangle\, dt$$

$$= \int (t + (t+1))\, dt$$

$$= \int (2t+1)\, dt$$

$$= \left[2 \cdot \frac{t^2}{2} + t \right]_0^1$$

$$= 1 + 1 = 2.$$

$$\int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt$$

$$= \int \langle t^2, t+1 \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int \left[t^2 + (t+1)(2t) \right] dt$$

$$= \int \left(t^2 + 2t^2 + 2t \right) dt$$

$$= \int \left(3t^2 + 2t \right) dt$$

$$= \left[3 \cdot \frac{t^3}{3} + 2 \cdot \frac{t^2}{2} \right]_0^1$$

$$= 1 + 1 = 2.$$ SAME ✓

In fact:

$$\int_{\text{curve}} \vec{F} = f(\text{end point}) - f(\text{start})$$

$$= f(1,1) - f(0,0)$$

$$= (1\cdot 1 + 1) - (0\cdot 0 + 0)$$

$$= 2.$$

That's why the two paths give the same answer.

Now let's change $\vec{F}$ a little bit

$$\vec{F}(x,y) = \langle y, x+1 \rangle$$
$$\vec{G}(x,y) = \langle y, 2x+1 \rangle$$

Integrate $\vec{G}$ along the two paths.

$$\int_0^1 \vec{G}(\underset{t,\,t}{\cancel{\vec{r}_1(t)}}) \cdot \underset{1,\,1}{\cancel{\vec{r}_1{}'(t)}}\, dt$$

$$= \int \langle t, 2t+1 \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int (t + (2t+1))\, dt$$

$$= \int (3t+1)\, dt$$

$$= \left(3\cdot\frac{t^2}{2} + t\right)\Big|_0^1$$

$$= \frac{3}{2} + 1 = \frac{5}{2}.$$

$$\int_0^1 \vec{G}\big(\underbrace{\cancel{\vec{r}_2(t)}}_{\langle t,\, t^2\rangle}\big) \circ \underbrace{\vec{r}_2{}'(t)}_{\langle 1,\, 2t\rangle}\, dt$$

$$= \int \langle t^2, 2t+1\rangle \bullet \langle 1, 2t\rangle\, dt$$

$$= \int \big(t^2 + (2t+1)(2t)\big)\, dt$$

$$= \int \left(t^2 + 4t^2 + 2t\right) dt$$

$$= \int \left(5t^2 + 2t\right) dt$$

$$= \left[5\cdot\frac{t^3}{3} + 2\cdot\frac{t^2}{2}\right]_0^1$$

$$= \frac{5}{3} + 1 = \frac{8}{3} \neq \frac{5}{2}$$

NOT THE SAME !

Today we'll discuss what went wrong.

But first, Kinetic Energy.

Consider a moving particle $\vec{r}(t)$ with mass m. Define

$$KE(t) = \underbrace{\frac{1}{2} m \|\vec{r}\,'(t)\|^2}_{\text{WHY ?}}$$

Suppose force field $\vec{F}$ acts on the particle, so $\vec{F}(\vec{r}(t)) = m\vec{r}\,''(t)$.

Compute $KE'(t)$.

$$\begin{aligned} KE(t) &= \frac{1}{2} m \|\vec{r}\,'(t)\|^2 \\ &= \frac{1}{2} m \underbrace{\vec{r}\,'(t) \cdot \vec{r}\,'(t)} \end{aligned}$$

Product Rule

$$KE'(t) = \frac{1}{2}m\left[\vec{r}''(t)\cdot\vec{r}'(t) + \vec{r}'(t)\cdot\vec{r}''(t)\right]$$

$$= \frac{1}{2}m\left[2\vec{r}''(t)\cdot\vec{r}'(t)\right]$$

$$= \underbrace{m\,\vec{r}''}(t)\cdot\vec{r}'(t)$$

$$= \underbrace{\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)}_{\text{What do we see ?}}.$$

$KE'(t)$ looks familiar !

$$KE(t) = \int \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

$$\int_{\text{curve}} \vec{\text{Force}} = KE(\text{end}) - KE(\text{start})$$

$$= \text{increase in } KE$$

Applies for ANY force field.

Now, assume $\vec{F}$ is conservative:

$$\vec{F} = -\nabla f \quad \text{for some } f.$$

Then we also have

$$\int_{\text{curve}} \vec{F} = \int -\nabla f$$

$$= - \int \nabla f$$

$$= - \left[f(\text{end}) - f(\text{start}) \right]$$

$$= f(\text{start}) - f(\text{end})$$

↑

Fund Thm Line Integrals

So let's define the potential energy

$$PE(t) = f(\vec{r}(t)).$$

Then combining the above equations:

$$KE(\text{end}) - KE(\text{start}) = PE(\text{start}) - PE(\text{end}).$$

$$KE(\text{start}) + PE(\text{start}) = KE(\text{end}) + PE(\text{end}).$$

"Conservation of Mechanical Energy"

Energy is converted between KE & PE but never destroyed.

This is <u>why</u> gradient vector fields are called "conservative".

Example: Gravity near planet.

$$\vec{F}(x,y,z) = \langle 0, 0, -mg \rangle$$

$$\vec{r}(0) = \langle 0,0,0 \rangle$$

$$\vec{r}'(0) = \langle 0,0,v \rangle \quad \text{up. } (v>0)$$

$$m\vec{r}''(t) = \vec{F}(\vec{r}(t))$$

$$m\vec{r}''(t) = \langle 0, 0, -mg \rangle$$

$$\vec{r}''(t) = \langle 0, 0, -g \rangle \quad \text{constant.}$$

$$\vec{r}'(t) = \langle 0, 0, -gt + v \rangle$$

$$\vec{r}(t) = \langle 0, 0, -\tfrac{1}{2}gt^2 + vt \rangle$$

$$KE(t) = \frac{1}{2}m\|\vec{r}'(t)\|^2$$

$$= \frac{1}{2}m\left[0^2+0^2+(-gt+v)^2\right]$$

$$= \frac{1}{2}m\left[g^2t^2-2gvt+v^2\right]$$

$$= \frac{1}{2}mg^2t^2 - mgvt + \frac{1}{2}mv^2.$$

Next: Observe that $\vec{F}$ is conservative.

$$f(x,y,z) = mgz$$

$$-\nabla f = \langle 0,0,-mg\rangle = \vec{F}.$$

Define

$$PE(t) = f(\vec{r}(t)).$$

$$= f(0,0,-\tfrac{1}{2}gt^2+vt)$$

$$= mg(-\tfrac{1}{2}gt^2+vt)$$

$$= -\frac{1}{2}mg^2t^2 + mgvt$$

Finally we have

$$KE(t) + PE(t) = \underbrace{\frac{1}{2}mv^2}_{\text{independent of } t.}$$

$$PE(\text{start}) = f(0,0,0) = 0$$

$$KE(\text{start}) = \frac{1}{2}m\|\vec{r}\,'(0)\|^2 = \frac{1}{2}mv^2$$

When the projectile reaches max height we get $\|\vec{r}\,'(t)\| = 0$, so $KE(\text{top}) = 0$.

$$PE(\text{top}) = \frac{1}{2}mv^2 - KE(\text{top})$$

$$PE(\text{top}) = \frac{1}{2}mv^2$$

$$\cancel{m}g\,z(\text{top}) = \frac{1}{2}\cancel{m}v^2$$

$$z(\text{top}) = \frac{1}{2g}v^2$$

This is the max height of the particle. Note: It is independent of mass!

UNITS:

$$g \sim \text{accel} \sim m/s^2$$

$$v \sim \text{velocity} \sim m/s$$

$$\frac{1}{2g} \cdot v^2 \sim \frac{1}{m/s^2} \cdot \left(\frac{m}{s}\right)^2 \sim m$$

So $\frac{1}{2g} v^2 \sim$ Length $\checkmark$

Back to Math.

Since $\vec{G} = \langle y, 2x+1 \rangle$ does not satisfy "independence of path", it cannot be a gradient vector field.

Is there an easier way to see this?

Theorem (Conservative Vector Fields).

Given vector field in $\mathbb{R}^2$:

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

The following statements are equivalent.

- $\vec{F} = \nabla f$ for some $f(x,y)$
- $\displaystyle\int_{\text{Loop}} \vec{F} = 0$ for any loop
- "Cross-Partial Property"

$$P_y = Q_x$$

Check : $\vec{F}(x,y) = \langle y, x+1 \rangle$

$$P(x,y) = y$$

$$Q(x,y) = x+1$$

$$P_y = 1$$

$$Q_x = 1$$

SAME

So $\vec{F}$ is conservative.

But $\vec{G}(x,y) = \langle y, 2x+1 \rangle$

$$P_y = 1$$

$$Q_x = 2$$

NOT SAME

So $\vec{G}$ is <u>not</u> conservative.

3D Version: Given

$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

The following are equivalent:

- $\vec{F} = \nabla f$ for some $f(x,y,z)$

- $\displaystyle\int_{Loop} \vec{F} = 0$ for any Loop.

- $\begin{cases} P_y = Q_x \\ P_z = R_x \\ Q_z = R_y \end{cases}$ "cross-partial property"

[In Higher Dimensions:

$\vec{F}(x_1, \ldots, x_n) = \langle F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n) \rangle$

Cross-Partial property says

$$\frac{dF_i}{dx_j} = \frac{dF_j}{dx_i} \text{ for all } i \neq j.$$

EASY TO CHECK :)]

Example: $\quad P \quad Q \quad R$

$$\vec{F}(x,y,z) = \langle 3x^2 z,\ z^2,\ x^3 + 2yz \rangle$$

Check cross partials:

$P_y = 0$ & $Q_x = 0$ ✓

$P_z = 3x^2$ & $R_x = 3x^2$ ✓

$Q_z = 2z$ & $R_y = 2z$ ✓

This guarantees that $\vec{F}$ has an antiderivative scalar field.

How can we find it?

TWO METHODS:

(1) Try really hard.

Looking for $f(x,y,z)$ such that

$$f_x(x,y,z) = 3x^2 z$$

$$f_y(x,y,z) = z^2$$

$$f_z(x,y,z) = x^3 + 2yz$$

START:

$$f_y = z^2$$

$$f = z^2 y + g(x,z)$$

$$f_x = 3x^2 z$$

$$f_x = 0 + g_x$$

$$g_x = 3x^2 z$$

$$g = x^3 z + h(y,z)$$

Seems like we're going around in circles!

(2) Use the Fund Thm:

If $\vec{F} = \nabla f$ then

$$\int_{\text{curve}} \vec{F} = f(\text{end}) - f(\text{start}).$$

(Independent of the shape of curve.).

TRICK: Fix some start point

start = $(0,0,0)$

Consider <u>any</u> path from $(0,0,0)$ to some point (a,b,c).

Say $\vec{r}(t) = (at, bt, ct)$
$t = 0$ to 1.

Then

$$\int_{\text{curve}} \vec{F} = \underbrace{f(a,b,c)}_{\substack{\text{this is}\\ \text{what we want}\\ \text{to know.}}} - \underbrace{f(0,0,0)}_{\text{const.}}.$$

So let's compute:

$$\int_0^1 \vec{F}(at, bt, ct) \cdot \langle a, b, c\rangle \, dt$$

$$= \int_0^1 \left\langle 3(at)^2(ct), (ct)^2, (at)^3 + 2(bt)(ct) \right\rangle \cdot \langle a, b, c\rangle \, dt.$$

$$= \int 3a^3ct^3 + bc^2t^2 + ca^3t^3 + 2bc^2t^2 \, dt$$

$$= 3a^3c\frac{t^4}{4} + bc^2\frac{t^3}{3} + ca^3\frac{t^4}{4} + 2bc^2\frac{t^3}{3} \Big|_0^1$$

$$= \underbrace{\frac{3}{4}a^3c + \frac{bc^2}{3} + \frac{ca^3}{4} + \frac{2bc^2}{3}}_{\text{This is our desired } f(a,b,c).}$$

In other words:

$$f(x,y,z) = \frac{3}{4}x^3z + \frac{1}{3}yz^2 + \frac{1}{4}x^3z + \frac{2}{3}yz^2.$$

$$= x^3z + yz^2$$

CHECK:

$$f(x,y,z) = x^3z + yz^2$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle 3x^2z, z^2, x^3 + 2yz \rangle$$

$$= \vec{F} \quad \checkmark$$

It worked.