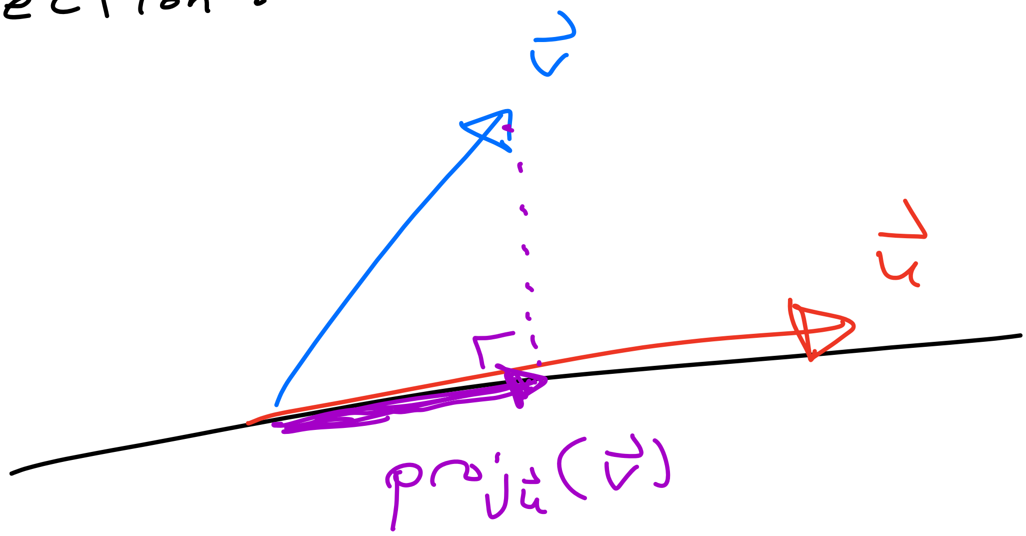
HW 5 is posted; due on Tues.

Quiz 5 next Wed.

Projection:



Formula:

$$\text{proj}_{\vec{u}}(\vec{v}) = \underbrace{\frac{\vec{u}\circ\vec{v}}{\vec{u}\circ\vec{u}}}_{\text{scalar}} \underbrace{\vec{u}}_{\text{vector}}$$

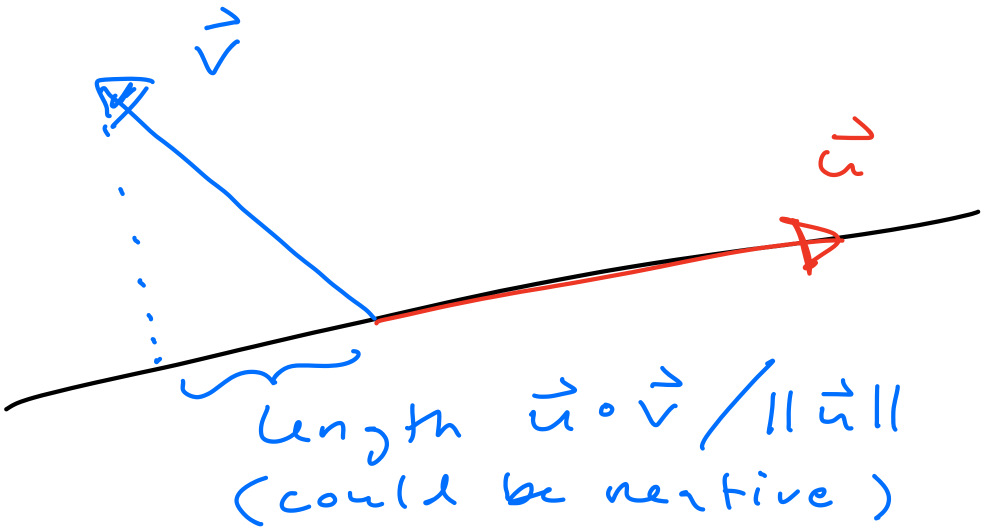
$$= \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|^2}\cdot\vec{u}$$

$$= \frac{\vec{u}\circ\vec{v}}{\|\vec{u}\|}\cdot\frac{\vec{u}}{\|\vec{u}\|}$$

a unit vector in the direction of $\vec{u}$.

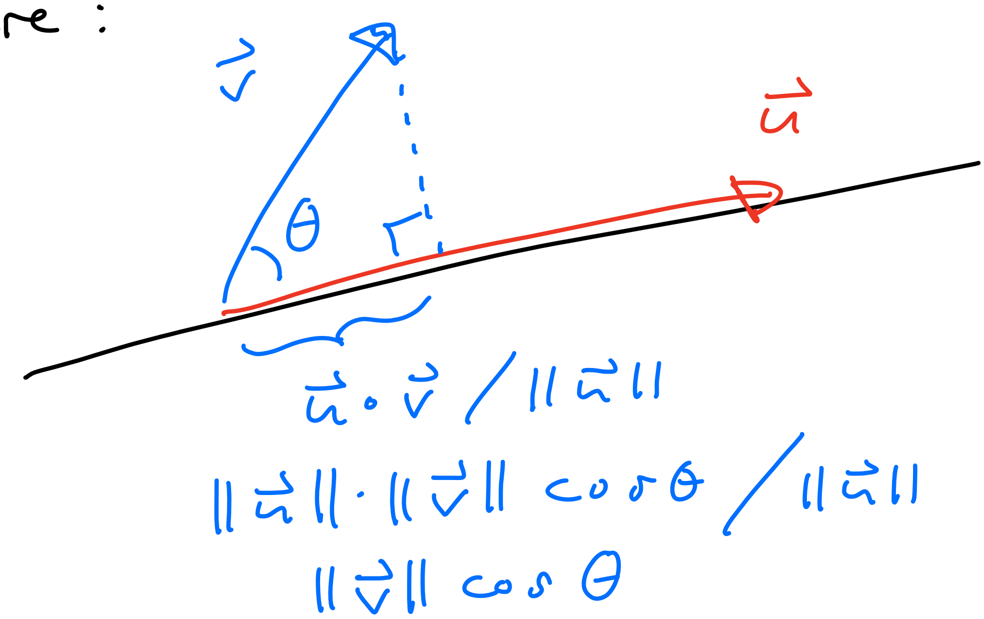
The magnitude (positive or negative) of the projection is $\vec{u} \cdot \vec{v} / \|\vec{u}\|$.

[Special case $\|\vec{u}\|$ is nicest.]



Call $\vec{u} \cdot \vec{v} / \|\vec{u}\|$ the "component of $\vec{v}$ in the direction of $\vec{u}$"

Picture:



Example: Project $\vec{v} = \langle 1, 0, 0 \rangle$
onto $\vec{u} = \langle 1, 1, 1 \rangle$

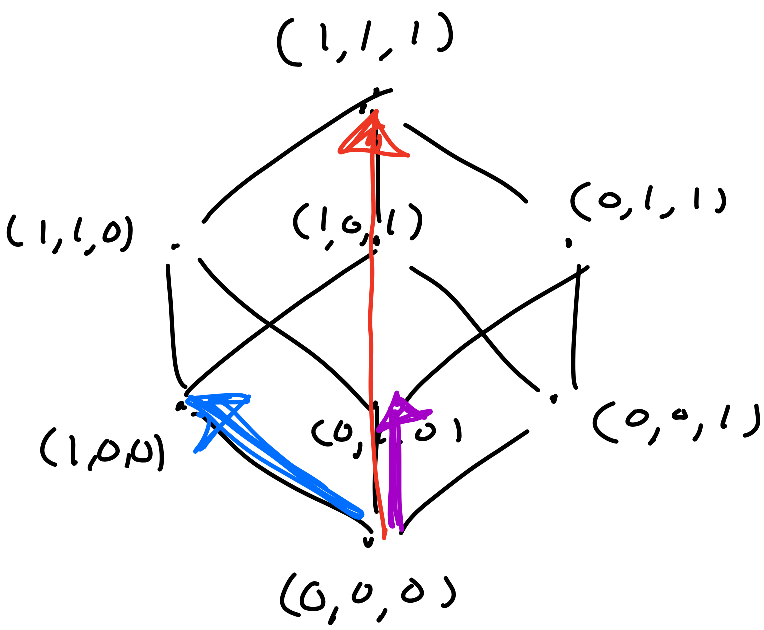
$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{1+0+0}{1+1+1} \langle 1, 1, 1 \rangle$$

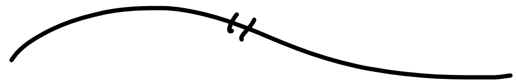
$$= \frac{1}{3} \langle 1, 1, 1 \rangle$$

$$= \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$$

Picture: A Cube sitting on corner.

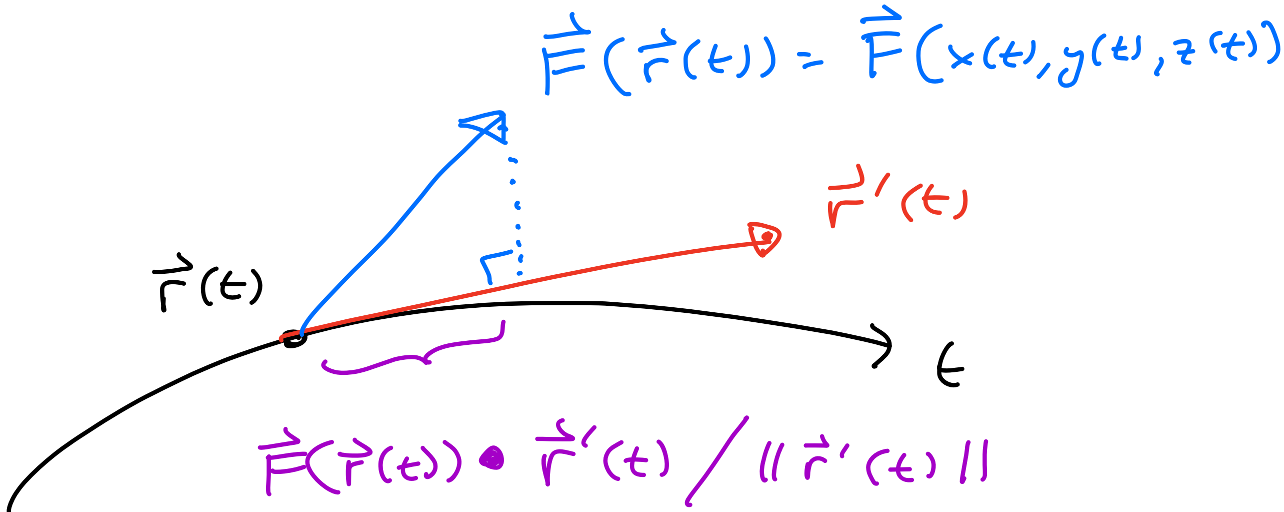


projection of (1,0,0) on (1,1,1) is $\frac{1}{3}$ of the way up the cube.



We use projection to define the integral of a vector field along a parametrized curve.

Consider vector field $\vec{F}(x,y,z)$ and curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.



$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \,/\, \|\vec{r}\,'(t)\|$

component of vector field $\vec{F}$ in direction of the curve $\vec{r}$.

Define integral of $\vec{F}$ along $\vec{r}$ as the integral of this component:

$$\int_{\text{curve}} \underbrace{\frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}_{\text{scalar}} \underbrace{\|\vec{r}\,'(t)\| \, dt}_{ds}$$

If we don't want to mention the parametrization, we can write

$$\int_C \vec{F} \cdot \underbrace{\vec{T}}_{\text{unit vector tangent to curve.}} \, ds$$

After parametrizing we get

$$\vec{T} = \underbrace{\vec{r}\,'(t) / \| \vec{r}\,'(t) \|}_{\text{unit vector tangent to curve.}}$$

$$ds = \underbrace{\| \vec{r}\,'(t) \|\, dt}_{\text{tiny piece of arc length}}$$

That's a lot of Jargon!

MEANING:

$$\int_C \vec{F} \cdot \vec{T} \, ds = \text{"how much (on average) does } \vec{F} \text{ point along the curve"?}$$

Physics:

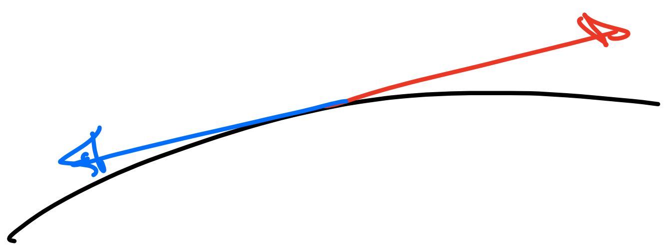
$$\int \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

= "how much work done on particle $\vec{r}(t)$ by force field $\vec{F}$"?

= "kinetic energy added to the particle by force field."

e.g. If $\vec{F}$ is friction then we always have $\vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t) < 0$

[force opposes the motion, i.e., is in the opposite direction from your velocity:



]

In this case

$$\int \underbrace{\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\text{always} < 0}\, dt < 0$$

This force <u>decreases</u> your KE.

Example: Gravity near surface of the Earth.

Pick coordinates so

z-axis points "up"

$z = 0$ is ground level.

Particle of mass m.

Launch the particle directly up with speed v.

$$\vec{r}(0) = \langle 0, 0, 0 \rangle$$

$$\vec{r}\,'(0) = \langle 0, 0, v \rangle.$$

$$\vec{r}\,''(t) = \langle 0, 0, -32 \text{ ft/sec}^2 \rangle.$$

Integrate:

$$\vec{r}\,'(t) = \langle \cancel{c_1}, \cancel{c_2}, -32t + \cancel{c_3} \rangle$$

0 0 v

$$\vec{r}\,'(t) = \langle 0, 0, -32t + v \rangle$$

$$\vec{r}(t) = \langle \cancel{c_4}, \cancel{c_5}, -16t^2 + vt + \cancel{c_6} \rangle$$

0 0 0

$$\vec{r}(t) = \langle 0, 0, -16t^2 + vt \rangle .$$

The force satisfies Newton's 2nd:

$$\vec{F}(t) = m\,\vec{r}\,''(t).$$

$$= m \langle 0, 0, -32 \rangle$$

$$= \underbrace{\langle 0, 0, -32m \rangle}_{\text{constant vector.}}$$

KEY Property of Gravity:

It has an anti-derivative, meaning if $\vec{F}(x,y,z)$ is the gravitational force, then we can find a scalar field $f(x,y,z)$

such that

$$\vec{F}(x,y,z) = \nabla f(x,y,z).$$

[Jargon: Vector field $\vec{F}$ with an anti-deriv $\vec{F} = \nabla f$ is called a "conservative vector field".]

For us:

$$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$$

at any point (x,y,z). Look for an anti-derivative $f(x,y,z)$.

$$\vec{F} = \nabla f$$

$$\langle 0, 0, -32m \rangle = \langle f_x, f_y, f_z \rangle.$$

$f_z = -32m \rightsquigarrow f = -32mz$ + something that does not involve z.

$$f(x,y,z) = -32mz + g(x,y)$$

for some function $g(x,y)$.

NEXT: $f_x = 0$.

$$\frac{d}{dx}\left(-32mz + g(x,y)\right) = 0$$

$$0 + g_x = 0$$

$$g_x = 0$$

$$g(x,y) = h(y).$$

for some function $h(y)$ of y.

Currently: $f(x,y,z) = -32mz + h(y)$.

FINALLY: $f_y = 0$.

$$\frac{d}{dy}\left(-32mz + h(y)\right) = 0$$

$$0 + h'(y) = 0$$

$$h(y) = c$$

for some constant c.

Conclusion: $f(x,y,z) = -32mz + c$.

For physical reasons, want

$$\vec{F} = -\nabla f$$

so take $f(x,y,z) = +32mz + c$.

If $\vec{F}$ is a force field

& $\vec{F} = -\nabla f$ then

f is called "potential energy".

In our case:

$$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$$

= force of gravity acting on a particle of mass m at point (x,y,z).

$$f(x,y,z) = +32mz + c$$

= gravitational potential of a particle of mass m at point (x,y,z).

Fundamental Theorem of "Line Integrals" (i.e. integrals of vector fields along curves).

$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$\int_{\text{along curve}} \nabla f = f(\text{endpoint}) - f(\text{start point})$$

e.g. $f(x,y,z) = xyz$

$$\nabla f(x,y,z) = \langle yz, xz, xy \rangle$$

Integrate along some curve:

$$\vec{r}(t) = \langle t, t^2, t^3 \rangle$$

$t = 1$ to $t = 2$.

Prediction: $\displaystyle\int_{\text{curve}} \nabla f = f(\vec{r}(2)) - f(\vec{r}(1)).$

$$= f(2,4,8) - f(1,1,1)$$

$$= 2 \cdot 4 \cdot 8 - 1 \cdot 1 \cdot 1 = 63.$$

Check:

$$\vec{F} = \nabla f = \langle yz, xz, xy \rangle.$$

$$\int_{\text{curve}} \vec{F} = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int \langle t^2 \cdot t^3, t \cdot t^3, t \cdot t^2 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \, dt$$

$$= \int_1^2 \left(t^5 + 2t^5 + 3t^5 \right) dt$$

$$= \int_1^2 6t^5 \, dt$$

$$= 6 \cdot \frac{1}{6} t^6 \Big|_1^2$$

$$= 2^6 - 2^1 = 63 \quad \checkmark$$

Proof of F.T.L.I.

Chain Rule:

$$\frac{d}{dt}\left(f(\vec{r}(t)) \right)$$

$$= \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t)$$

Integrate both sides with resp. to t.

Let $g(t) = f(\vec{r}(t))$.

$$\int \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t)\,dt.$$

$$= \int_a^b \frac{d}{dt} g(t)\,dt$$

Calc I

$$= g(b) - g(a) \quad \checkmark$$

Physics: Let $\vec{F}$ be force field. Suppose $\vec{F} = -\nabla f$ for some scalar field f (called the "potential energy". Then

$$\underbrace{\underbrace{\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt}_{\text{increase in KE}} = \underbrace{-f(\vec{r}(b)) + f(\vec{r}(a))}_{\text{decrease in PE.}}}_{\text{conservation of mechanical energy.}}$$

Back to our example:

$$\vec{F}(x,y,z) = \langle 0, 0, -32m \rangle$$

$$f(x,y,z) = +32mz + c$$

Let's choose c so potential

energy is zero on the ground.

$\rightsquigarrow c = 0.$

$f(x,y,0) = 0.$

$$PE(t) = f(\vec{r}(t))$$

$$= f(0,0,-16t^2+vt)$$

$$= +32m(-16t^2+vt)$$

$$= -512mt^2+32mvt$$

Define the Kinetic energy at time t:

$$KE(t) = \frac{1}{2} m \|\text{velocity}\|^2$$

$$= \frac{1}{2} m \|\vec{r}\,'(t)\|^2$$

$$= \frac{1}{2} m \|\langle 0, 0, -32t+v\rangle\|^2$$

$$= \frac{1}{2} m (-32t+v)^2$$

$$= \frac{1}{2} m (1024t^2 - 64vt + v^2)$$

$$= 512mt^2 - 32mvt + \frac{1}{2}mv^2$$

Conclusion:

$$KE(t) + PE(t) = \frac{1}{2}mv^2$$

constant, i.e., independent of t.

At time $t = 0$ we have

$$KE(0) = \frac{1}{2}mv^2$$

$$PE(0) = 0$$

When the particle reaches the top, it has no velocity, so $KE(\text{top}) = 0$.

Hence

$$KE(\text{top}) = 0$$

$$PE(\text{top}) = \frac{1}{2}mv^2.$$

$$+32mz = \frac{1}{2}mv^2$$

$$z = \frac{1}{64}v^2$$

This is how high the particle will go. We could have solved this by maximizing the z coord:

$$z(t) = -16t^2 + vt.$$

But I wanted to illustrate the concept of potential energy, which applies in much more general situations.