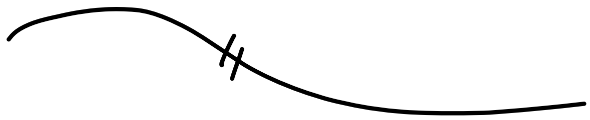
Next week:

- No class on Mon
- HW5 due Tues
- Quiz 5 on Wed

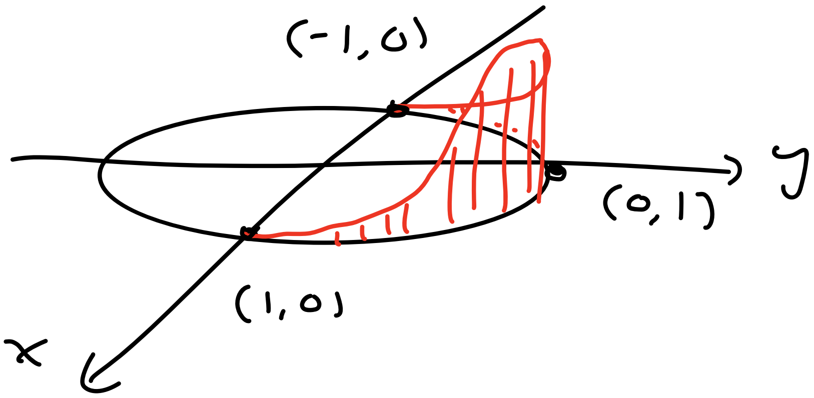
} moved one class later



Last Time: Integrating a scalar field f over a curve in $\mathbb{R}^2$, $\mathbb{R}^3$ or a surface in $\mathbb{R}^3$.

$$\int_{\text{curve}} \overbrace{f \, \underbrace{ds}_{\text{tiny piece of arc length on curve}}}^{\text{tiny piece of mass, or area, ...}}$$

e.g. Find the area of vertical wall above circle $x^2+y^2=1$ in x,y-plane & below parabolic surface $z=x^2$, with $y \geq 0$. Picture:



Area of wall = $\int$ area of skinny rectangles

$$= \int x^2 \, ds$$

height   length of base.

To compute this we parametrize the base curve:

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$
$$t = 0 \text{ to } \pi.$$

According to definition:

$$\int f \, ds = \int f(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

In our case:

$$f(x,y) = x^2$$

$$\vec{r}(t) = \langle \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

So area of wall

$$= \int_0^{\pi} (\cos t)^2 \cdot 1 \, dt$$

[Trig Identity?

$$\cos(2t) = \cos^2 t - \sin^2 t$$

$$\cos(2t) = \cos^2 t - (1 - \cos^2 t)$$

$$\cos(2t) = 2\cos^2 t - 1$$

$$2\cos^2 t = \cos(2t) + 1$$

$$\cos^2 t = \frac{1}{2}(1 + \cos(2t)).$$

]

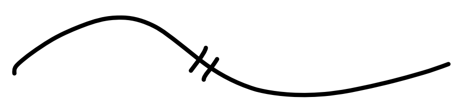
$$= \int_0^{\pi} \cos^2 t \, dt$$

$$= \int_0^{\pi} \frac{1}{2}\left(1 + \cos(2t)\right) dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t) \right]_0^{\pi}$$

$$= \frac{1}{2}\Big[\pi + \frac{1}{2}\overset{0}{\cancel{\sin(2\pi)}} - 0 - \frac{1}{2}\underset{0}{\cancel{\sin(0)}} \Big]$$

$$= \pi/2$$



Integrate scalar field over a surface in $\mathbb{R}^3$

$$\iint_{\text{surface}} \overbrace{f \, \underbrace{dA}_{\substack{\text{tiny piece of area} \\ \text{in the surface}}}}^{\text{tiny piece of mass}}$$

Typical : Surface area

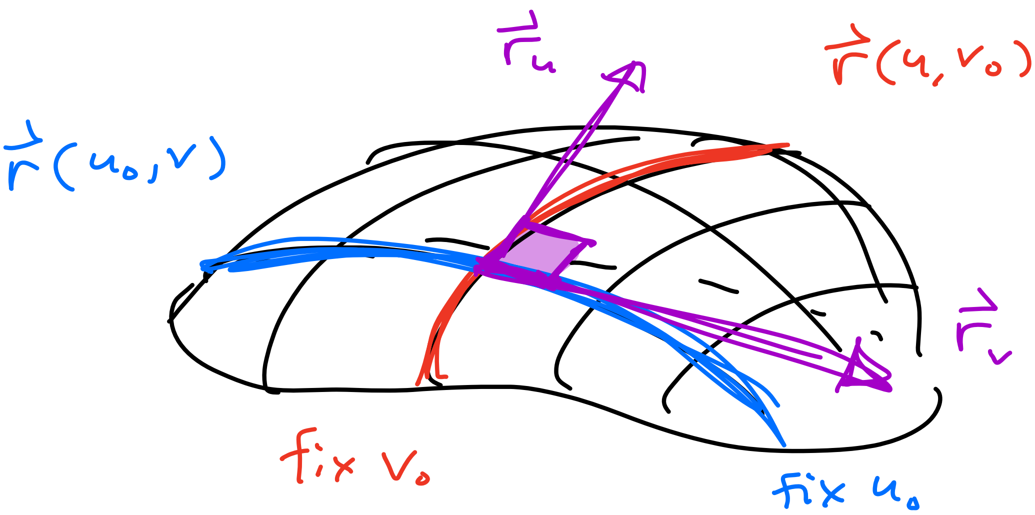
$$\iint_{\text{surface}} 1 \, dA$$

add up all the tiny pieces of area.

How to compute ?

Parametrize the surface :

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



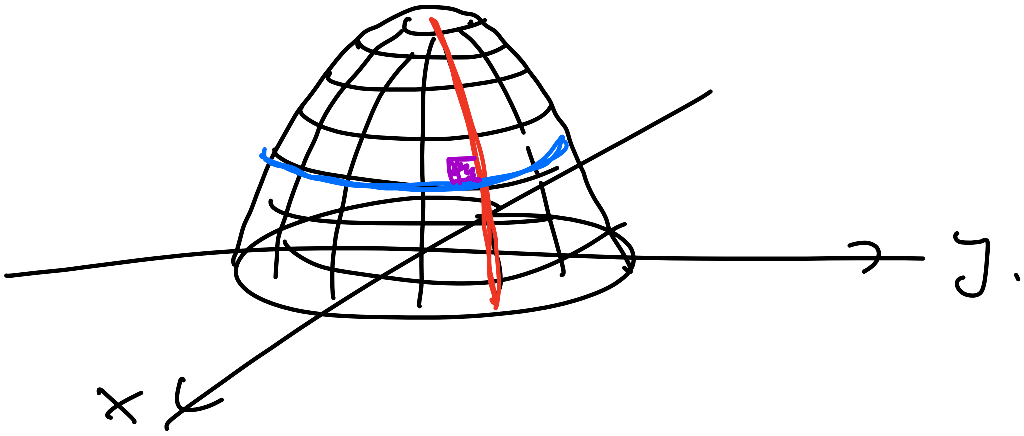
dA = area of tiny parallelogram

$$= \| \vec{r}_u \times \vec{r}_v \| \, du \, dv$$

So surface area is

$$\iint 1\, dA = \iint \| \vec{r}_u \times \vec{r}_v \| \, du\, dv.$$

Example: Surface area of parabolic dome $z = 1 - x^2 - y^2$, $x^2 + y^2 \le 1$.



Parametrize the surface:

Use Polar in x,y-plane.

$$x = u \cos v \qquad 0 \le u \le 1$$
$$y = u \sin v \qquad 0 \le v \le 2\pi$$
$$[x^2 + y^2 = u^2]$$

$$z = 1 - x^2 - y^2$$
$$= 1 - u^2$$

$$\vec{r}(u,v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, -2u \rangle$$

$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 2u^2\cos v, 2u^2 \sin v, u\cancel{\cos^2 v} + u\cancel{\sin^2 v} \rangle$$

$$= \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^4 \cancel{\cos^2 v} + 4u^4 \cancel{\sin^2 v} + u^2}$$

$$= \sqrt{4u^4 + u^2}$$

$$= \sqrt{u^2(4u^2+1)}$$

$$= u\sqrt{4u^2+1}$$

So surface area

$$= \iint \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

$$= \iint u\sqrt{4u^2+1} \, du \, dv$$

$$= \int_0^{2\pi} dv \int_0^1 u\sqrt{4u^2+1}\, du$$

LUCKY !

$$= 2\pi \int_0^1 u\sqrt{4u^2+1}\, du$$

$w = 4u^2+1$

$dw = 8u\, du$

$u\, du = \frac{1}{8} dw$

$$= 2\pi \int_1^5 \frac{1}{8}\sqrt{w}\, dw$$

$$= \frac{\pi}{4}\left[\frac{2}{3} w^{3/2}\right]_1^5$$

$$= \frac{\pi}{6}\left[5^{3/2} - 1\right]$$

DONE.

General Pattern:

Parametrized k-dim "surface" living in n-dim space has form

$$\vec{r}(u_1, \dots, u_k) = \langle x_1(u_1, \dots, u_k), x_2(u_1, \dots, u_k), \dots, x_n(u_1, \dots, u_k) \rangle$$

Jacobian Matrix

$$J = \underbrace{\begin{pmatrix} (x_1)_{u_1} & \cdots & (x_1)_{u_k} \\ \vdots & & \vdots \\ (x_n)_{u_1} & \cdots & (x_n)_{u_k} \end{pmatrix}}_{k \text{ cols}} \Bigg\} n \text{ rows}$$

To compute "k-volume" of this region:

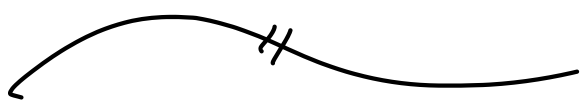
$$\int_{\text{region}} \underbrace{\sqrt{\det(J^T J)}}_{\substack{k\text{-volume} \\ \text{stretch factor}}} \, du_1 \, du_2 \cdots du_k$$

This is how the pattern continues:

$$\int_{\text{curve}} \|\vec{r}_u\| \, du$$

$$\iint_{\text{surface}} \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

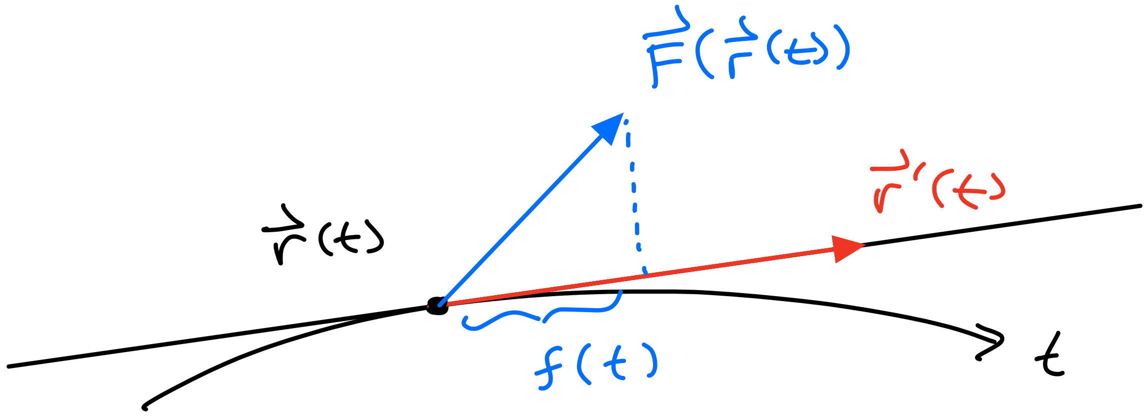
etc.



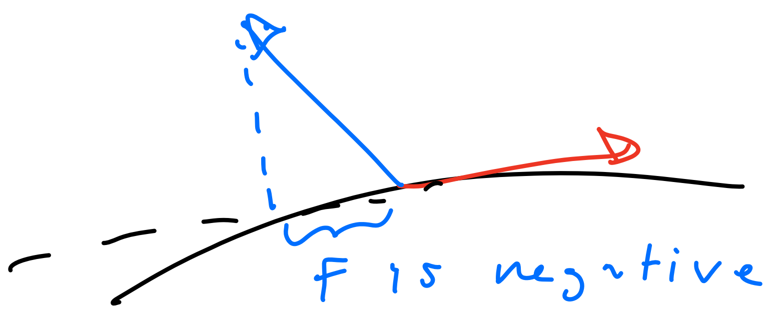
Next: Integrate vector fields over curves and surfaces.

WHY? Physics (Energy).

Think of particle moving in a force field (e.g. gravity).



Let $f(t)$ be the component of the force $\vec{F}(\vec{r}(t))$ in the direction of the velocity $\vec{r}\,'(t)$, so $f(t)$ is a scalar. It can be negative when force opposes motion:



From physics

$$\int_{\text{curve}} f \, ds$$

$$\int_{\text{curve}} f(t) \, \|\vec{r}\,'(t)\| \, dt$$

= amount of kinetic energy added to particle by force field.

Could be negative. Friction always resists the motion, so $f(t) < 0$

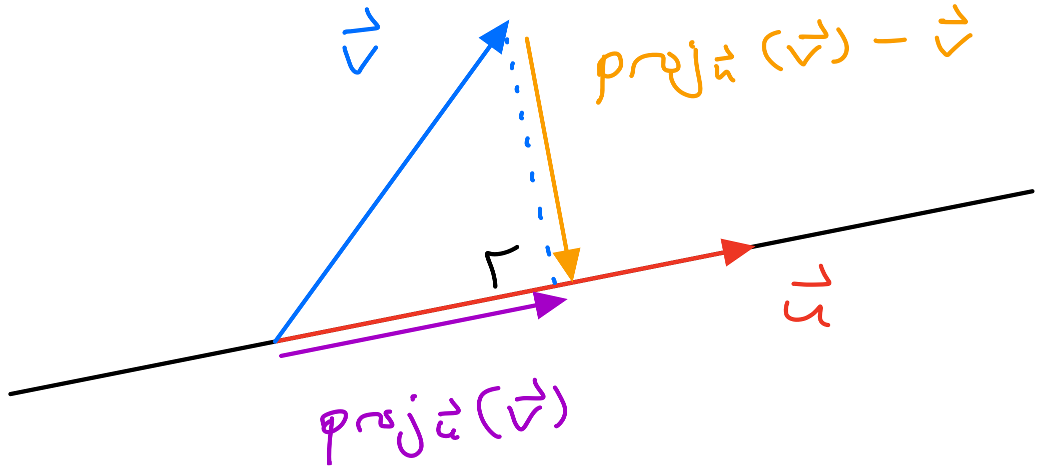
Change in Kinetic energy due to Friction $= \int \underbrace{f(t)\,\|\vec{r}\,'(t)\|}_{\text{always} < 0}\,dt < 0.$

Let's be precise. Consider a force field & parametrized curve:

$$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\vec{r} : \mathbb{R} \longrightarrow \mathbb{R}^3$$

We need a formula for the component of $\vec{F}(\vec{r}(t))$ in the direction of $\vec{r}\,'(t)$.

Projection:



Formula? TWO FACTS:

① $\mathrm{proj}_{\vec{u}}(\vec{v}) = \alpha \vec{u}$ for some scalar α.

② There is a right angle, i.e., the dot product of two vectors is zero. Which two vectors?

$$\vec{u} \cdot \left(\mathrm{proj}_{\vec{u}}(\vec{v}) - \vec{v} \right) = 0.$$

Put ① & ② together:

$$\vec{u} \cdot \left(\alpha \vec{u} - \vec{v} \right) = 0$$

$$\alpha\, \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} = 0$$

$$\alpha = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}$$

Conclusion:

$$\boxed{\operatorname{proj}_{\vec{u}}(\vec{v}) = \underbrace{\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}}_{\text{scalar}} \; \underbrace{\vec{u}}_{\text{vector}}}$$

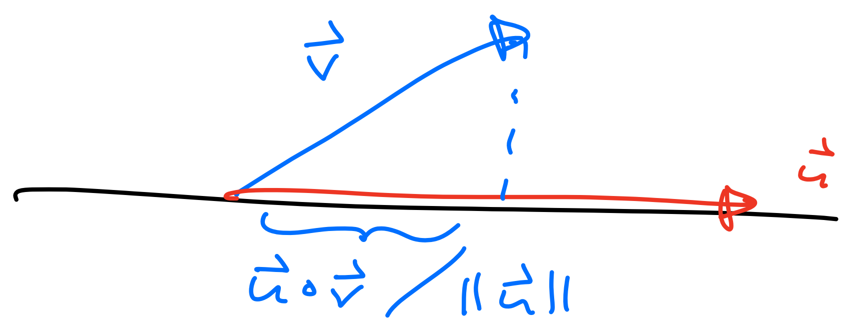
The length of the projection?

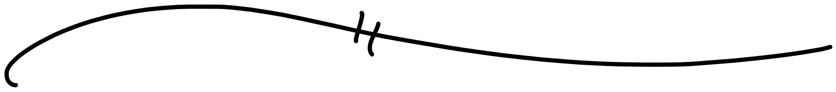
$$\|\operatorname{proj}_{\vec{u}}(\vec{v})\| = \left|\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right| \|\vec{u}\|$$

$$= \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\|^2} \|\vec{u}\|$$

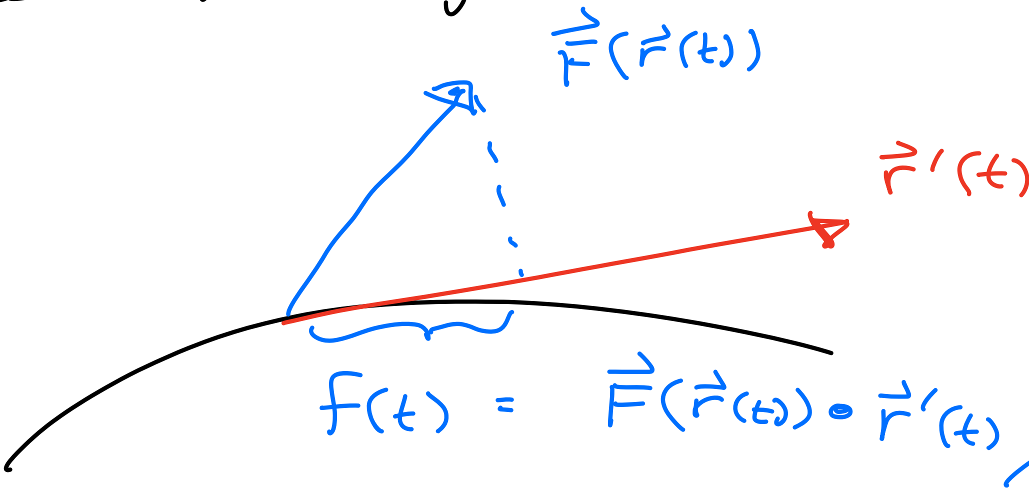
$$= |\vec{u} \cdot \vec{v}| \,/\, \|\vec{u}\|$$

If we want to allow negatives:





Force & Velocity:



$$f(t) = \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \Big/ \|\vec{r}'(t)\|$$

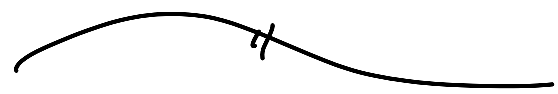
Finally: Work done by a changing force on a moving particle

$$= \int_{\text{curve}} f(t) \, \|\vec{r}'(t)\| \, dt$$

$$= \int_{\text{curve}} \frac{\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)}{\cancel{\|\vec{r}'(t)\|}} \cdot \cancel{\|\vec{r}'(t)\|} \, dt$$

$$= \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int (\text{force}) \cdot (\text{velocity})\, dt$$



Most Interesting Example:

Gravity.

More generally, say a force field

$$\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$$

is "conservative" if it is the gradient of some scalar field:

$$\vec{F} = \nabla f$$

$$\left(\vec{F} = -\nabla f \quad \text{in physics} \right).$$

We will see that conservative force fields have many special properties.

One property is the fundamental

Theorem:

If $\vec{F} = \nabla f$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \circ \vec{r}\,'(t)\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof: "Completely easy"

Chain rule:

$$[f(\vec{r}(t))]' = \nabla f(\vec{r}(t)) \bullet \vec{r}\,'(t).$$

Use Fundamental Theorem from Calc I & II:

dot product.

$$\int_a^b \vec{F}(\vec{r}(t)) \bullet \vec{r}\,'(t)\, dt$$

Calc I

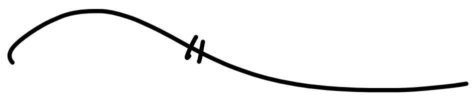
$$= \int_a^b [f(\vec{r}(t))]'\, dt$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

[Recall: For any $g(t)$ we have

$$\int_a^b g'(t)\,dt = g(b) - g(a).$$

Here we take $g(t) = f(\vec{r}(t))$.]



Interpretation:

Kinetic Energy + Potential Energy = Constant.