This week : Chapter 6

Next week :

- Mon no class
- ~~Quiz 5~~ HW 5 due on Tuesday
- wed "~~bonus lecture~~" Quiz 5
- Thurs & Fri no class.
- Final Project due Fri.

Chap 6 :

- Integrating over curves & surfaces
- vector field definitions
  (divergence & curl)
- "Fundamental Theorems"

Calc I & II : $\int f' = f$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

New : $\int \nabla f = f$

$$\iint_{\text{surface}} \nabla \times F = \int_{\substack{\text{boundary}\\ \text{curve}}} F$$

Stokes / Green's Theorem

$$\iiint_{\text{solid}} \nabla \cdot F = \iint_{\substack{\text{boundary}\\ \text{surface}}} F$$

Divergence Theorem

Don't have time to discuss these in detail.

Integrate along a curve. WHY?

Consider a wire C in 3D.

$\rho(x,y,z)$ = mass density of the wire at point (x,y,z)

= mass / unit length.

The mass of the wire:

$$\text{mass} = \int_C \overbrace{\rho\, \underbrace{ds}_{\text{tiny piece of length}}}^{\text{tiny piece of mass}}$$

To compute this we need to parametrize the curve. Say

$$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

for $a \le t \le b$.

[Here "t" is not really time; it's just a name for parameter.]

Then we define:

$$\int_C \rho \, ds = \int_a^b \rho(\vec{r}(t)) \underbrace{\|\vec{r}\,'(t)\|}_{\text{tiny piece of length}} dt$$

tiny mass at point $\vec{r}(t)$

Special Case: $\rho = 1$ then

arc length = mass

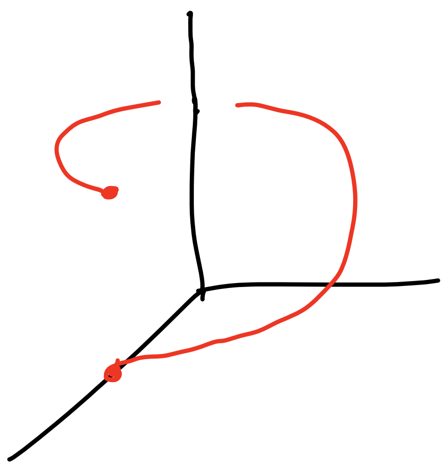
$$= \int_a^b 1 \, \|\vec{r}\,'(t)\| \, dt$$

as we already know!

Example: Consider helix

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$
$$0 \le t \le 2\pi.$$

Suppose density $\rho(x,y,z) = x^2 + y^2 + z$.



$$\text{mass of wire} = \int_C \rho \, ds$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1^2} = \sqrt{2}$$

Density at point $\vec{r}(t)$:

$$\rho(\vec{r}(t)) = \overset{x^2}{(\cos t)^2} + \overset{y^2}{(\sin t)^2} + \overset{z}{(t)}$$

$$= 1 + t \quad \text{nice} \quad \ddot\smile$$

$$\text{mass} = \int_0^{2\pi} \rho(\vec{r}(t)) \, \|\vec{r}\,'(t)\| \, dt$$

$$= \int_0^{2\pi} (1+t)\sqrt{2} \, dt$$

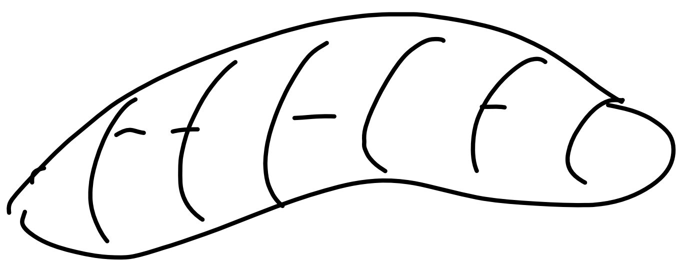
$$= \sqrt{2}\left[t + \frac{1}{2}t^2 \right]_0^{2\pi}$$

$$= \sqrt{2}\left[(2\pi) + \frac{1}{2}(2\pi)^2 \right].$$

DONE.

Integrating over a surface in $\mathbb{R}^3$.

WHY? Let D be 2D region living in 3D



$\rho(x,y,z) =$ Mass density / area

$$\text{mass of } D = \iint_D \overbrace{\rho \, \underbrace{dA}_{\text{tiny piece of area}}}^{\text{tiny piece of mass.}}$$

Special Case:

$$\text{surface area of } D = \iint_D 1 \, dA .$$

HOW TO COMPUTE ?

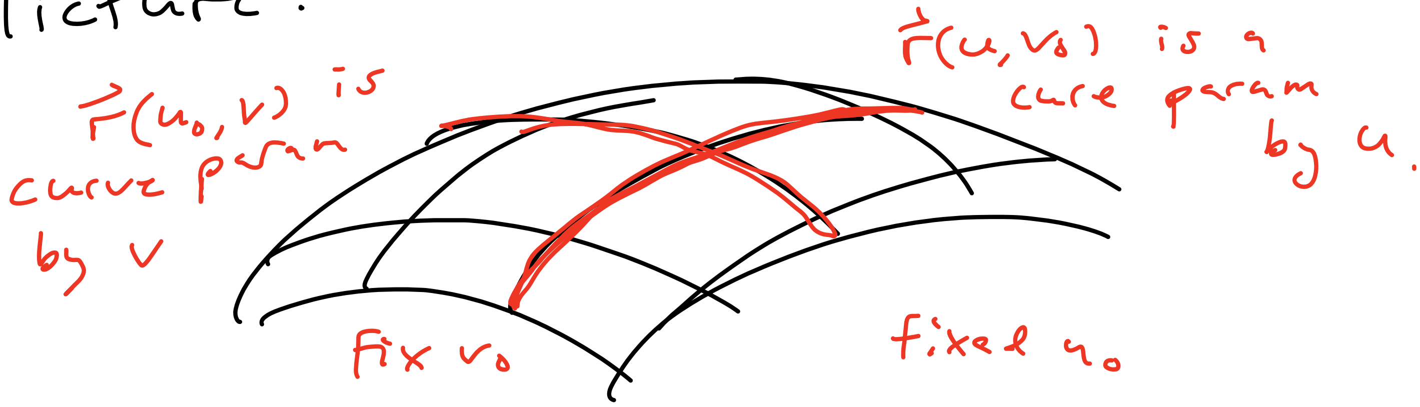
Need to parametrize the surface.

Can think of this as a function

$$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Picture:



IDEA:

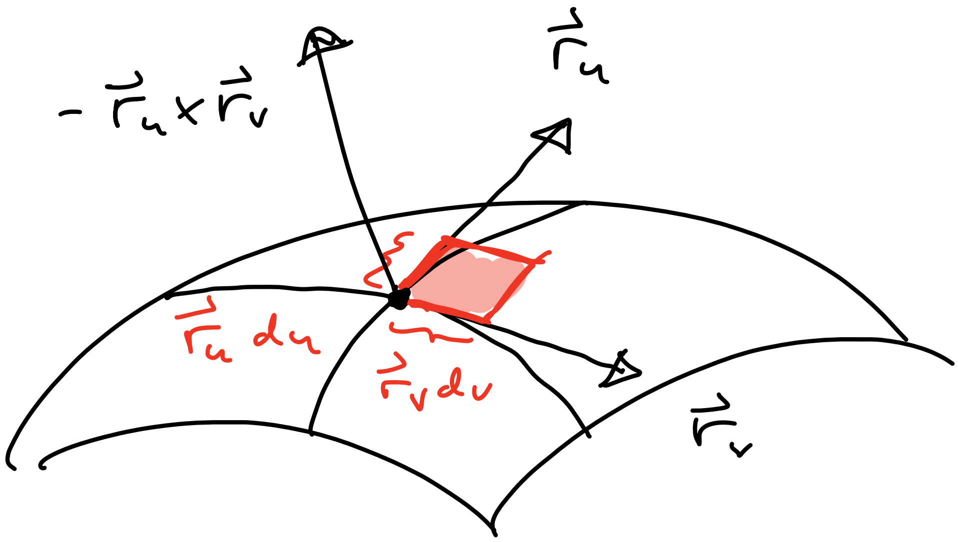
$$\text{mass} = \iint_D \rho(\vec{r}(u,v))\, du\, dv$$

NO! Need some kind of "Jacobian stretch factor"

Different ways to explain this.

Here's the most intuitive:

TWO "VELOCITY VECTORS"



Area of tiny parallelogram

scalars

$$dA = \|(\vec{r}_u\, du) \times (\vec{r}_v\, dv)\|$$

$$= \underbrace{\|\vec{r}_u \times \vec{r}_v\|}\, du\, dv$$

"Jacobian stretch factor"

[More Highbrow:

$$J_{\vec{F}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

strech factor

$$= \sqrt{\det\left(J_{\vec{F}}^{T} J_{\vec{F}}\right)} \, . \quad]$$

$$\text{mass} = \iint_D \rho(\vec{r}(u,v)) \, \|\vec{r}_u \times \vec{r}_v\| \, du\, dv \, .$$

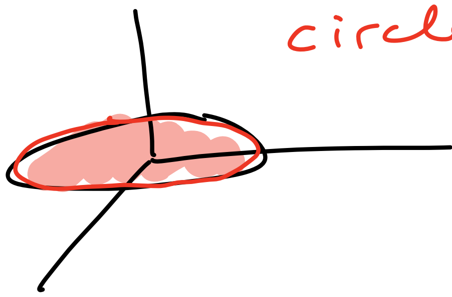
$$\text{surface area} = \iint_D 1 \, \|\vec{r}_u \times \vec{r}_v\| \, du\, dv \, .$$

Test: Area of a Circle in xy-plane

$$\vec{r}(u,v) = \langle u\cos v, \; u\sin v, \; 0 \rangle$$

$$0 \le u \le a \quad \& \quad 0 \le v \le 2\pi$$

circle radius a



$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, 0, u\cos^2 v + u\sin^2 v \rangle$

$= \langle 0, 0, u \rangle$

$\| \vec{r}_u \times \vec{r}_v \| = u \qquad (u \geq 0)$

surface area $= \iint \| \vec{r}_u \times \vec{r}_v \| \, du \, dv$

$= \iint u \, du \, dv$

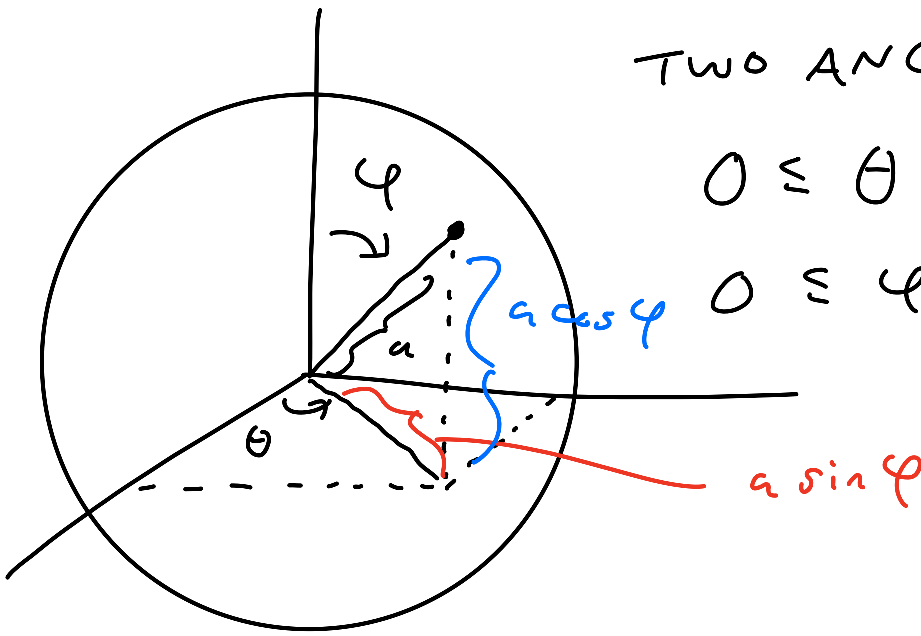
$= \int_0^{2\pi} dv \int_0^a u \, du$

$$= 2\pi \left[\frac{1}{2} a^2 \right]_0^a$$

$$= \pi a^2 \quad \checkmark \quad \text{area of circle.}$$

More interesting: Surface area of a sphere of radius a.

Parametrization ?



TWO ANGLES:

$$0 \le \theta \le 2\pi$$

$$0 \le \varphi \le \pi$$

Cartesian ?

$$x = a \sin\varphi \cos\theta$$
$$y = a \sin\varphi \sin\theta$$
$$z = a \cos\varphi$$

Parametrization:

$$\vec{r}(\theta,\varphi) = \langle x(\theta,\varphi), y(\theta,\varphi), z(\theta,\varphi)\rangle$$

$$= \langle a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi\rangle$$

$$\vec{r}_\theta = \langle -a\sin\theta\sin\varphi,\ a\cos\theta\sin\varphi,\ 0\rangle$$

$$\vec{r}_\varphi = \langle a\cos\theta\cos\varphi,\ a\sin\theta\cos\varphi,\ -a\sin\varphi\rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi =$$

$$\langle -a^2\cos\theta\sin^2\varphi,\ a^2\sin\theta\sin^2\varphi,$$

$$-a^2\sin^2\theta\sin\varphi\cos\varphi - a^2\cos^2\theta\sin\varphi\cos\varphi\rangle$$

$$-a^2\sin\varphi\cos\varphi$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{a^4\cancel{\cos^2\theta}\sin^4\varphi + a^4\cancel{\sin^2\theta}\sin^4\varphi + a^4\sin^2\varphi\cos^2\varphi}$$

$$= \cdots = a^2\sin\varphi$$

surface area $= \iint\limits_{D} \| \vec{r}_\theta \times \vec{r}_\varphi \| \, d\theta \, d\varphi$

$$= \iint a^2 \sin\varphi \, d\theta \, d\varphi$$

$$= a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi$$

$$= 2\pi a^2 \left[-\overset{1}{\cancel{\cos(\pi)}} + \overset{+1}{\cancel{\cos}(0)} \right]$$

$$= 4\pi a^2 \quad \checkmark$$

surface area of a sphere of radius a.