HW 4 due now.

Today: HW 4 Discussion
Quiz 4 Review.

Problem 1: Integrate $f(x,y) = 6x^2y$ over rectangle $R = [-1,1] \times [0,4]$.

$$\iint_R f\,dA = \iint_R 6x^2y\,dx\,dy$$

x First:

$$\int_0^4 \left(\int_{-1}^{1} 6x^2 y\,dx \right) dy$$

constant

$$= \int_0^4 \left[6 \cdot \frac{x^3}{3} \cdot y \right]_{-1}^{1} dy$$

$$= \int_0^4 \left[6 \cdot \frac{1}{3} \cdot y + 6 \cdot \frac{1}{3} \cdot y \right]$$

$$= \int_0^4 4y\,dy$$

$$= \left[4 \cdot \frac{1}{2} y^2 \right]_0^4$$

$$= 4 \cdot \frac{1}{2} (4)^2 - 0$$

$$= 2 \cdot 16 = 32 .$$

Also note $6x^2y$ is "separable":

$$\iint 6x^2y \, dx \, dy$$

$$= 6 \cdot \int x^2 \, dx \cdot \int y \, dy$$

Problem 2: Polar Coords.

$x = r\cos\theta$ $\qquad$ $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad$ $\theta = \arctan(y/x)$

Done $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ ✓

Now $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \det \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}$

$$r_x = \frac{d}{dx}\sqrt{x^2+y^2}$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2}(2x+0)$$

$$= x \Big/ \underbrace{\sqrt{x^2+y^2}}_{r}$$

Similar:

$$r_y = y \Big/ \underbrace{\sqrt{x^2+y^2}}_{r}$$

$$\theta_x = \frac{d}{dx}\arctan(y/x)$$

$$= \frac{1}{(\frac{y}{x})^2+1} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{x^2}{\cancel{x^2}} \frac{1}{(\frac{y}{\cancel{x}})^2+\cancel{x}^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{-y\,\cancel{x^2}}{x^2+y^2} \cdot \frac{1}{\cancel{x^2}}$$

$$= -y \Big/ (x^2+y^2)$$

$$\theta_y = \frac{d}{dy}\arctan\left(\frac{y}{x}\right)$$

$$= \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \cdot \frac{1}{x}$$

$$= \frac{x^2}{\cancel{x^2}} \, \frac{1}{\left(\frac{y}{\cancel{x}}\right)^2 + \cancel{1}\,x^2} \cdot \frac{1}{x}$$

$$= \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x}$$

$$= x \Big/ (x^2 + y^2) \, .$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \det \begin{pmatrix} x\big/\sqrt{x^2+y^2} & y\big/\sqrt{x^2+y^2} \\ -y\big/(x^2+y^2) & x\big/(x^2+y^2) \end{pmatrix}$$

$$= \frac{x^2}{(x^2+y^2)^{3/2}} - \frac{-y^2}{(x^2+y^2)^{3/2}}$$

$$= (x^2+y^2) \Big/ (x^2+y^2)^{3/2}$$

$$= (x^2+y^2)^{-1/2}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{r}$$

Verify:

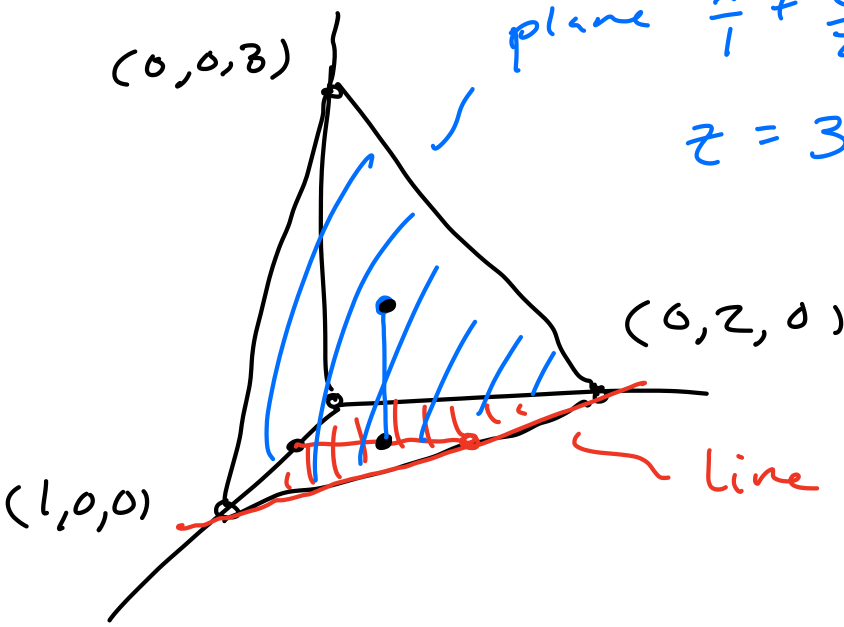
$$\underbrace{\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1}\ \checkmark$$

Knowing this we could have avoided the computation of $\partial(r,\theta)/\partial(x,y)$.

Meaning:

$$dx\,dy = \underbrace{r}_{\text{area stretch factor.}}\,dr\,d\theta\,.$$

Problem 3: Tetrahedron.



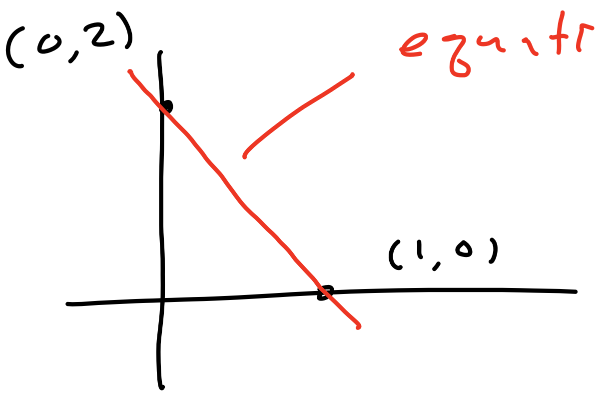
plane $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$

$z = 3\left(1 - x - y/2\right)$

line in x, y plane.

Fix $0 \leq x \leq 1$

Then $0 \leq y \leq 2(1-x)$



equation $\frac{x}{1} + \frac{y}{2} = 1$

$y = 2(1-x)$

$$0 \leq z \leq 3\left(1 - x - y/2\right)$$

$$\text{Volume} = \iiint 1 \, dV$$

$$= \iiint 1 \, dx\, dy\, dz$$

$$= \int_0^1 \left(\int_0^{2(1-x)} \left(\int_0^{3(1-x-y/2)} 1 \, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{2(1-x)} 3(\underbrace{1-x}_{\text{constant}} - \tfrac{y}{2}) \, dy \right) dx$$

$$= 3 \int_0^1 \left[(1-x)y - \frac{1}{2} \cdot \frac{1}{3} y^3 \right]_0^{2(1-x)} dx$$

$$= 3 \int_0^1 \left(2(1-x)^2 - \frac{1}{6} 8 (1-x)^3 \right) dx$$

$$= 3 \left[-2 \frac{(1-x)^3}{3} \cdot \frac{1}{+1} + \frac{8}{6} \cdot \frac{(1-x)^4}{4} \cdot \frac{1}{+1} \right]_0^1$$

$$= 3 \left[-0 + 0 - \left(-\frac{2}{3} \cdot 1 + \frac{2}{6} \cdot 1 \right) \right]$$

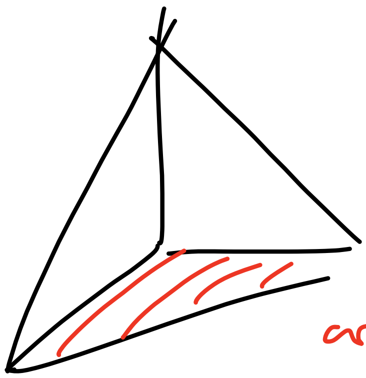
$$= 6\left(\frac{1}{3} - \frac{1}{6}\right)$$

$$= 6 \cdot \frac{1}{6} = 1$$

Another Way:

Volume of any cone in 3D is

$$\frac{1}{3}(\text{area of base})(\text{height})$$



height $= 3$

area of base $= 1 \cdot 2/2 = 1$

$$\text{volume} = \frac{1}{3} \cdot (1) \cdot (3) = 1$$

Remark: Tetrahedron with vertices

$(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$

with $a,b,c > 0$ has volume

$$\frac{1}{6} abc.$$

Proof: Do it for $a = b = c = 1$.
Then use "stretching" argument
as in Problem 5(b).

Problem 5:

(a) Volume of unit sphere

$$= \iiint_{\text{sphere}} 1 \, dV$$

spherical

$$= \iiint \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, d\varphi \int_0^1 \rho^2 \, d\rho$$

$$= 2\pi \left[-\cos(\pi) + \cos(0) \right] \cdot \frac{1}{3} (1)^3$$

$$= 2\pi \left[1 + 1 \right] \frac{1}{3}$$

$$= \frac{4}{3}\pi$$

Now consider ellipsoid.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

"u, v, w - substitution"

$$u = x/a \quad , \quad v = y/b \quad , \quad w = z/c$$

$$u^2 + v^2 + w^2 = 1 \qquad \ddot\smile$$

$$dx\,dy\,dz = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

$$= \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

e.g. $x = au \rightsquigarrow \begin{array}{l} x_u = a \\ x_v = 0 \\ x_w = 0 \end{array}$ Nice!

$$= a \det\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$$

$$- \cancel{0 \det\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}}$$

$$+ \cancel{0 \det\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}}$$

$$= abc.$$

Hence

$$\text{vol} = \iiint_{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \le 1} 1 \, dx\,dy\,dz$$

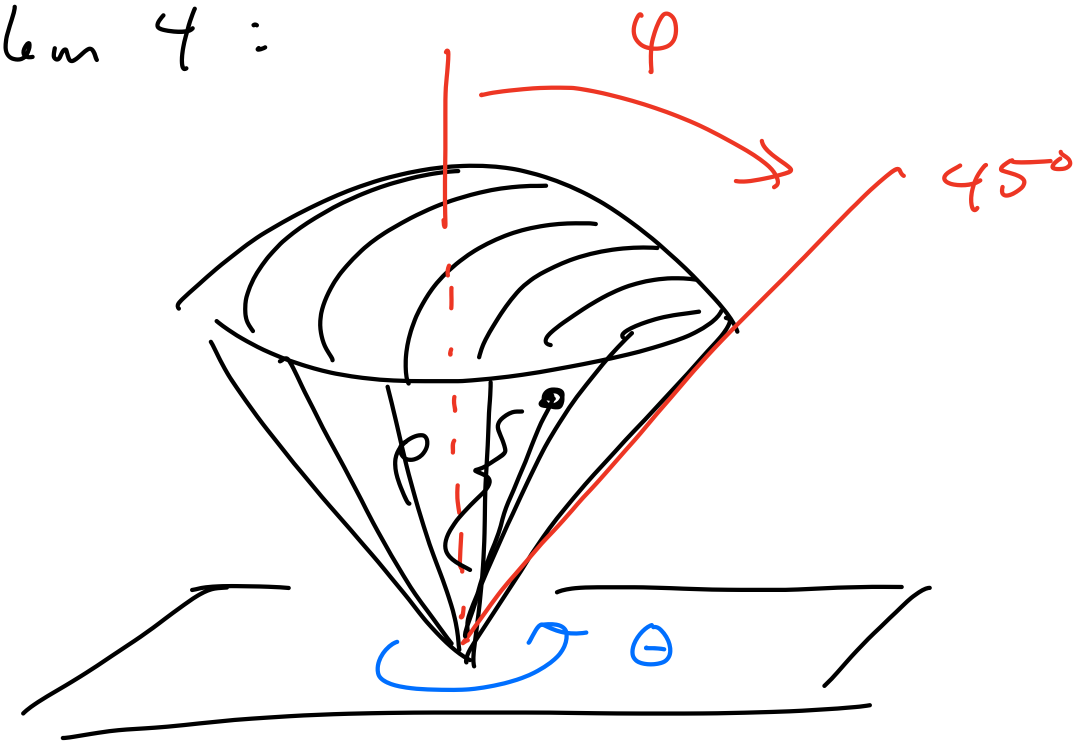
$$= \iiint_{u^2+v^2+w^2 \le 1} abc \, du\,dv\,dw$$

$$= abc \iiint_{u^2+v^2+w^2 \le 1} 1 \, du\,dv\,dw$$

$$= abc \,(\text{vol of unit sphere})$$

$$= abc \,\frac{4}{3}\pi \,.$$

Problem 4 :



$0 \le \varphi \le \pi/4$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 1 \,.$

density $= \dfrac{1 \text{ unit mass}}{\text{unit vol.}}$

mass $m = \iiint 1 \, dV$

$$= \iiint 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^1 \rho^2 \, d\rho$$

$$= 2\pi \left[-\cos\left(\frac{\pi}{4}\right) + \cos(0) \right] \left[\frac{1}{3}(1)^3 \right]$$

$$= \frac{2}{3}\pi \left[-\frac{\sqrt{2}}{2} + 1 \right]$$

$$= \frac{\pi}{3} \left(2 - \sqrt{2} \right)$$

Center of mass (Definition):

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right).$$

M_{yz} = moment about yz-plane

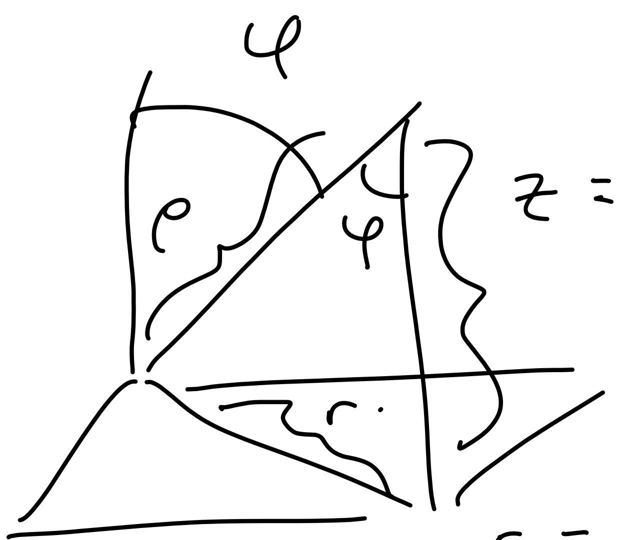
$$= \iiint (\text{dist to } yz \text{ plane}) \cdot (\text{density}) \, dV$$

$$= \iiint x \cdot 1 \cdot dV$$

$$= \iiint \, ? \;\; \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

[Spherical Coords:

$$\left.\begin{aligned} x &= r\cos\theta \\ r &= \rho\sin\varphi \end{aligned}\right\} x = \rho\sin\varphi\cos\theta$$

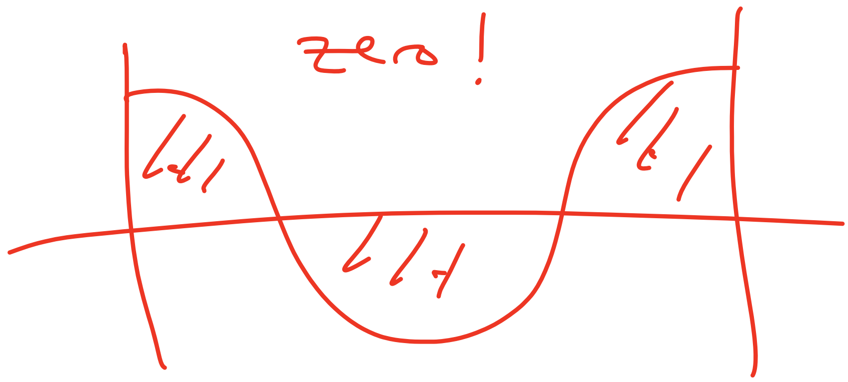


$$z = \rho\cos\varphi$$

$$r = \rho\sin\varphi \quad]$$

$$= \iiint \rho\sin\varphi\cos\theta \cdot \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \cos\theta \, d\theta \cdot \int_0^{\pi/4} \sin^2\varphi \, d\varphi \cdot \int_0^1 \rho^3 \, d\rho$$



$= 0$

As it should be.

Region is symmetric around the z-axis so center of mass is on z-axis, so $\bar{x}, \bar{y} = 0$

so $M_{yz} = 0$ & $M_{xz} = 0$.

$$M_{xy} = \iiint z \, dV$$

$$= \iiint \rho \cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \cos\varphi \sin\varphi \, d\varphi \int_0^1 \rho^3 \, d\rho$$

$$= \cancel{2}\pi \cdot \frac{1}{\cancel{4}_2} \cdot \int_0^{\pi/4} \frac{1}{2} \sin(2\varphi)\, d\varphi.$$

$$= \frac{\pi}{4} \left[-\frac{1}{2}\cos(2\varphi) \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} \left[-\frac{1}{2}\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}} + \frac{1}{2}\overset{1}{\cancel{\cos(0)}} \right]$$

$$= \pi / 8$$

Finally:

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{\cancel{\pi}/8}{\frac{\cancel{\pi}}{3}(2-\sqrt{2})}\right)$$

$$= \left(0, 0, \frac{3}{8} \cdot \frac{1}{2-\sqrt{2}}\right)$$

$$= (0, 0, 0.64)$$

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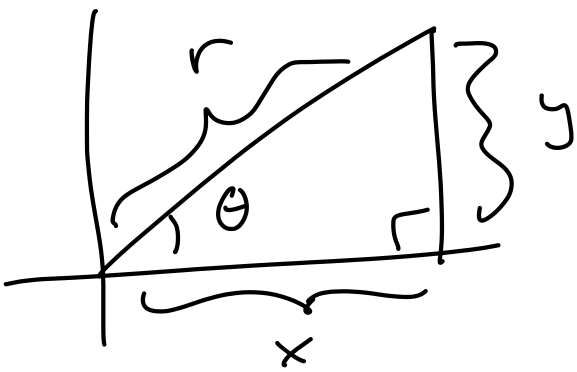
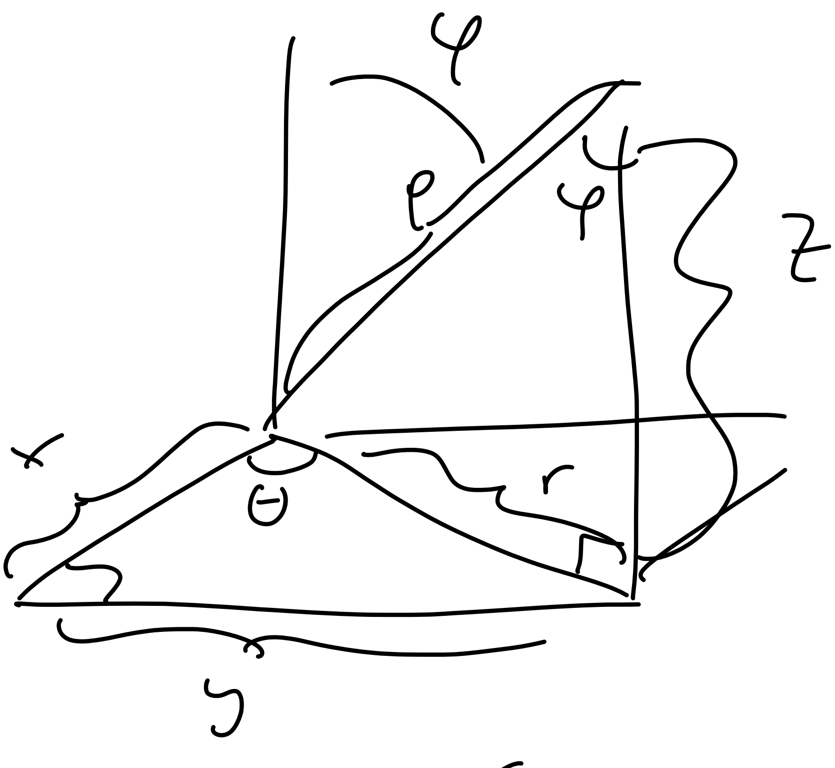
Quiz 4 Review:

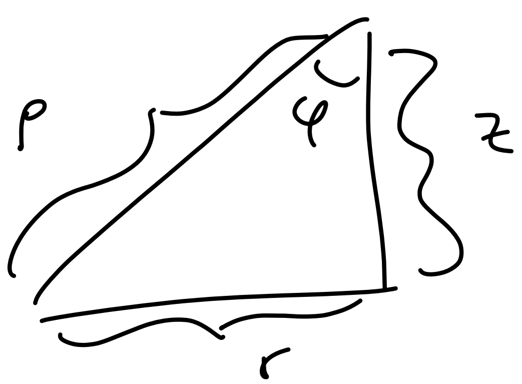


$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2 + y^2$

$r = \sqrt{x^2+y^2}$

$y/x = \tan\theta \rightsquigarrow \theta = \arctan(y/x)$





$$z = \rho \cos \varphi$$

$$r = \rho \sin \varphi$$

$$x = r \cos \theta = \rho \sin \varphi \cos \theta$$

$$y = r \sin \theta = \rho \cos \varphi \cos \theta .$$

Jacobians:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2 \sin \varphi$$

The hardest thing to remember!

$$dx\,dy = r\,dr\,d\theta$$

$$dx\,dy\,dz = \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

Center of Mass

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m} \right)$$

$m = \iiint \text{density } dV$

$M_{yz} = \iiint x \cdot \text{density } dV$

$M_{xz} = \iiint y \cdot \text{density } dV$

$M_{xy} = \iiint z \cdot \text{density } dV.$

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

$dA = dx\,dy$

$m = \iint \text{density } dA$

$M_y = \iint x \cdot \text{density } dA$

$M_x = \iint y \cdot \text{density } dA.$