Problem 1. Surface Area. Compute the area of the parametrized surface

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle = \langle u, u, v^2\rangle$$

for $0 \leq u \leq 1$ and $0 \leq v \leq 1$.

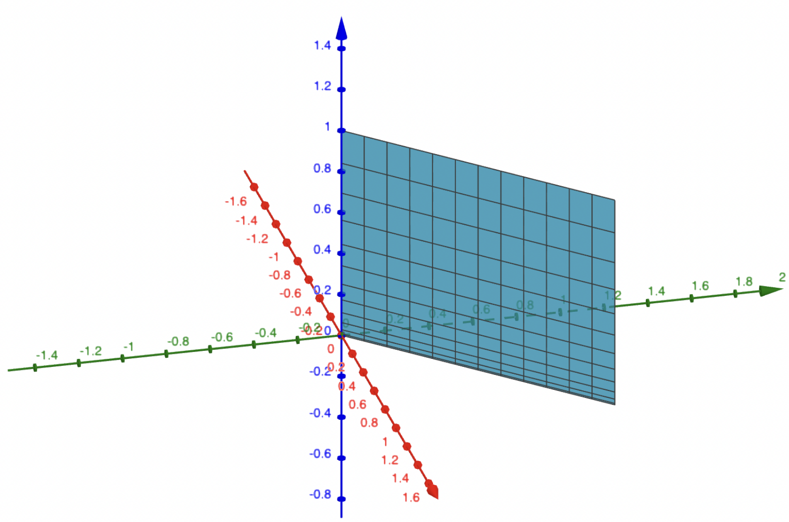
First we compute the stretch factor:

$$\begin{aligned}
\mathbf{r}_u &= \langle 1, 1, 0\rangle, \\
\mathbf{r}_v &= \langle 0, 0, 2v\rangle, \\
\mathbf{r}_u \times \mathbf{r}_v &= \langle 2v, -2v, 0\rangle, \\
\|\mathbf{r}_u \times \mathbf{r}_v\| &= \sqrt{4v^2 + 4v^2} \\
&= \sqrt{8v^2} \\
&= 2\sqrt{2}v \text{ because } v \geq 0.
\end{aligned}$$

Then we compute the area:

$$\begin{aligned}
\iint 1\, dA &= \iint \|\mathbf{r}_u \times \mathbf{r}_v\|\, dudv \\
&= \iint 2\sqrt{2}v\, dudv \\
&= 2\sqrt{2} \cdot \int_0^1 du \cdot \int_0^1 v\, dv \\
&= 2\sqrt{2}(1)(1/2) \\
&= \sqrt{2}.
\end{aligned}$$

Remark: It's really hard to find a surface whose area is computable by hand. This surface is secretly just a rectangle with base $\sqrt{2}$ and height 1:



Problem 2. Line Integrals. Integrate the vector field[1] $\mathbf{F}(x,y) = \langle -y, x\rangle$ around the unit circle $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \leq t \leq 2\pi$.

From the definition we have

$$\begin{aligned}
\int_{\text{circle}} \mathbf{F} \bullet \mathbf{T} &= \int \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt \\
&= \int \mathbf{F}(\cos t, \sin t) \bullet \langle -\sin t, \cos t\rangle\,dt \\
&= \int \langle -\sin t, \cos t\rangle \bullet \langle -\sin t, \cos t\rangle\,dt \\
&= \int \left[\sin^2 t + \cos^2 t\right]\,dt \\
&= \int_0^{2\pi} 1\,dt \\
&= 2\pi.
\end{aligned}$$

Remark: Since this integral is not zero, we conclude that the vector field $\mathbf{F}$ is not conservative.

Problem 3. Conservative Vector Fields. Find a scalar field $f(x,y)$ such that

$$\nabla f(x,y) = \langle 2x + 2y, 2x + 2y\rangle.$$

We will use the Fundamental Theorem of Line Integrals (or whatever you want to call it). Consider the path $\mathbf{r}(t) = \langle xt, yt\rangle$ for t from 0 to 2π. Then we have

$$\begin{aligned}
f(x,y) - f(0,0) &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\
&= \int_0^1 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt \\
&= \int_0^1 \langle 2xt + 2yt, 2xt + 2yt\rangle \bullet \langle x, y\rangle\,dt \\
&= \int_0^1 \left[(2xt + 2yt)x + (2xt + 2yt)y\right]\,dt \\
&= \int_0^1 (2x^2 + 4xy + 2y^2)t\,dt \\
&= (2x^2 + 4xy + 2y^2) \cdot \int_0^1 t\,dt \\
&= (2x^2 + 4xy + 2y^2) \cdot (1/2) \\
&= x^2 + 2xy + y^2.
\end{aligned}$$

We conclude that $f(x,y) = x^2 + 2xy + y^2$, plus an arbitrary constant. Check:

$$\begin{aligned}
(x^2 + 2xy + y^2)_x &= 2x + 2y + 0, \\
(x^2 + 2xy + y^2)_y &= 0 + 2x + 2y.
\end{aligned}$$

[1]That is, integrate the component of $\mathbf{F}$ that is tangent to the curve. You know, the usual thing.