**Math 211** **Summer 2022**
**Quiz 4** **Drew Armstrong**

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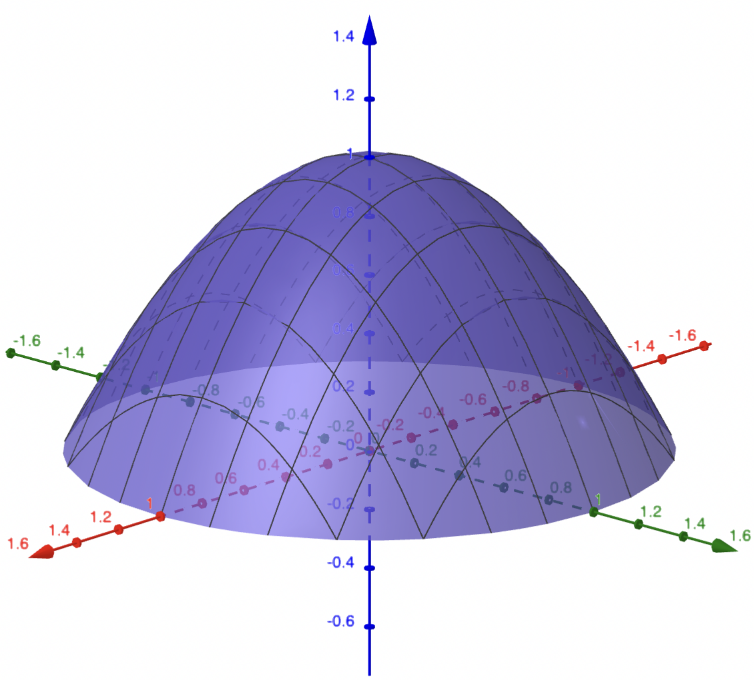
**Problem 1. Double Integrals.** Use polar coordinates to integrate the scalar function $f(x, y) = 1 - x^2 - y^2$ over the unit disk $x^2 + y^2 \leq 1$.

Polar coordinates are defined by $x = r\cos\theta$ and $y = r\sin\theta$, with $dxdy = rdrd\theta$. This is a good choice because the function $f$ and the domain of integration both have rotational symmetry. To be specific, we have $f = 1 - x^2 - y^2 = 1 - r^2$, and the domain is parametrized by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Hence the integral is

$$\begin{aligned}
\iint_{\text{disk}} f &= \iint_{\text{disk}} (1 - r^2) r dr d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 (r - r^3)\, dr \\
&= 2\pi \cdot \left[\frac{1}{2}r - \frac{1}{4}r^4\right]_0^1 \\
&= 2\pi \left[\frac{1}{2} - \frac{1}{4}\right] \\
&= \frac{\pi}{2}.
\end{aligned}$$

Remark: If we want, we could interpret this as the volume between the $z$-axis and the parabolic dome $z = 1 - x^2 - y^2$:



**Problem 2. Triple Integrals.**

(a) Find a parametrization for the tetrahedron in $\mathbb{R}^3$ with vertices

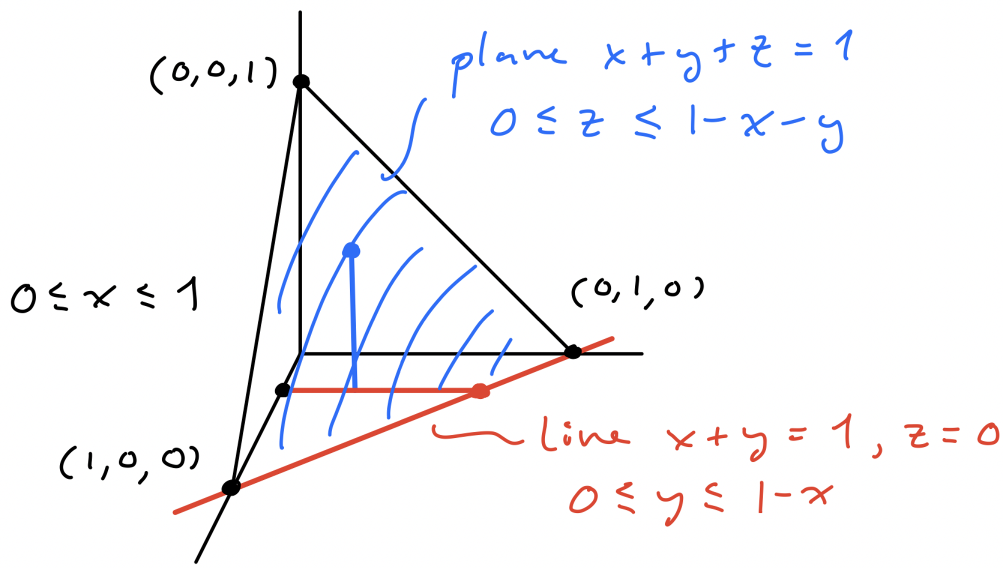
$$(0,0,0), \quad (1,0,0), \quad (0,1,0), \quad \text{and} \quad (0,0,1).$$

(b) Use your parametrization to compute the volume of the tetrahedron.

(a): If we choose $x$, then $y$, then $z$, we obtain the following parametrization:

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1, \\ 0 & \leq & y & \leq & 1-x, \\ 0 & \leq & z & \leq & 1-x-y. \end{array}$$

Here is a picture:




(b): The volume is

$$\begin{aligned} \iiint_{\text{tetrahedron}} 1\,dxdydz &= \int_0^1 \left( \int_0^{1-x} \left( \int_0^{1-x-y} dz \right) dy \right) dx \\ &= \int_0^1 \left( \int_0^{1-x} (1-x-y)\,dy \right) dx \\ &= \int_0^1 \left[ (1-x)y - \frac{1}{2}y^2 \right]_0^{1-x} dx \\ &= \int_0^1 \left( (1-x)(1-x) - \frac{1}{2}(1-x)^2 \right) dx \\ &= \int_0^1 \frac{1}{2}(1-x)^2\,dx \\ &= \left[ -\frac{1}{6}(1-x)^3 \right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$