Math 211 **Summer 2022**
Quiz 3 **Drew Armstrong**

Problem 1. Chain Rule. Let $f(x,y) = xe^y$ with $x(u,v) = uv$ and $y(u,v) = u^2v$. Use the multivariable chain rule to compute df/du.

The chain rule says that

$$\frac{df}{du} = \frac{df}{dx}\cdot\frac{dx}{du} + \frac{df}{dy}\cdot\frac{dy}{du}, \quad \text{or} \quad f_u = f_x \cdot x_u + f_y \cdot y_u.$$

In order to compute this we compute the partial derivatives:

$$\begin{aligned} f_x &= e^y, \\ f_y &= xe^y, \\ x_u &= v, \\ y_u &= 2uv. \end{aligned}$$

Then we put them together:

$$f_x = ve^y + 2uvxe^y.$$

If we want, we can express the answer in terms of u and v:[1]

$$f_x = ve^{u^2v} + 2uv(uv)e^{u^2v}.$$

Problem 2. Tangent Plane. Let $f(x,y,z) = x^2y - z$. Find the equation of the tangent plane to the surface $f(x,y,z) = 1$ at the point $(x_0,y_0,z_0) = (2,1,3)$.

The equation of the tangent plane to the surface $f(x,y,z) = \text{constant}$ at a point (x_0,y_0,z_0) is

$$\begin{aligned} \nabla f(x_0,y_0,z_0) \bullet \langle x-x_0, y-y_0, z-z_0\rangle &= 0, \\ \langle f_x(x_0,y_0,z_0), f_x(x_0,y_0,z_0), f_x(x_0,y_0,z_0)\rangle \bullet \langle x-x_0, y-y_0, z-z_0\rangle &= 0, \\ f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) &= 0. \end{aligned}$$

In our case we have

$$\begin{aligned} f_x &= 2xy, \\ f_y &= x^2, \\ f_z &= -1, \end{aligned}$$

and $(x_0,y_0,z_0) = (2,1,3)$, so the tangent plane is

$$\begin{aligned} f_x(2,1,3)(x-2) + f_y(2,1,3)(y-2) + f_z(2,1,3)(z-3) &= 0 \\ 2(2)(1)(x-2) + (2)^2(y-1) - 1(z-3) &= 0 \\ 4(x-2) + 4(y-1) - 1(z-3) &= 0 \\ 4(x-2) + 4(y-1) - 1(z-3) &= 0 \\ 4x + 4y - z &= 9. \end{aligned}$$

Here is a picture:

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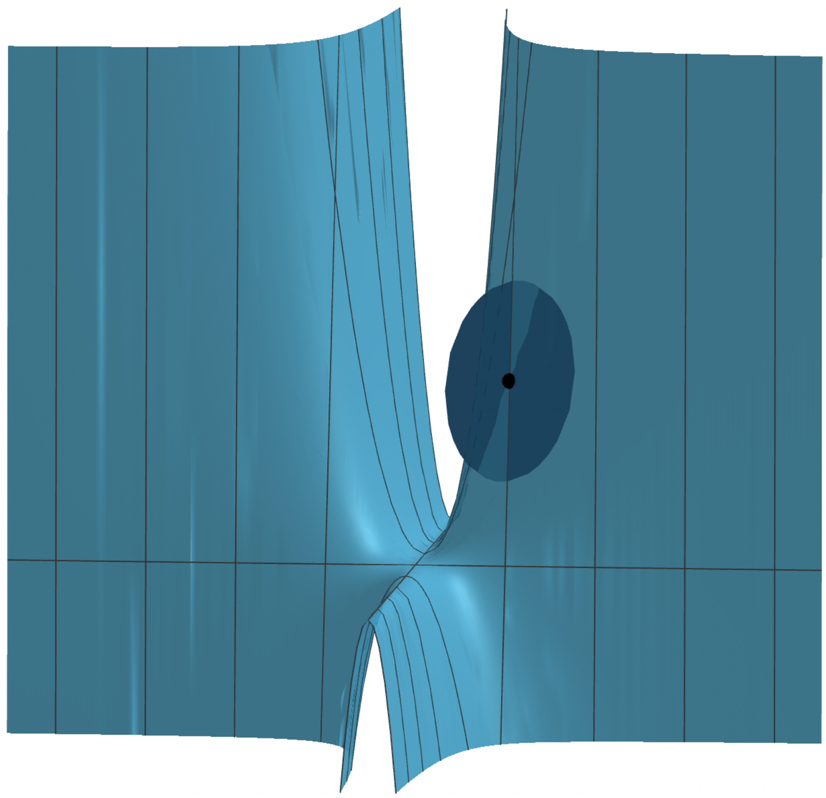
[1] And, I guess, we could simplify it.



Problem 3. Optimization. The scalar field $f(x,y) = x^3 - y^3 + xy$ has two critical points: $(0,0)$ and $(1/3,-1/3)$. Use the second derivative test to determine whether each of these is a local maximum or minimum, saddle point, or degenerate.

We need to compute the determinant of the Hessian matrix. We begin by computing the second partial derivatives:[2]

$$\begin{aligned} f_x &= 3x^2 + y, \\ f_y &= -3y^2 + x, \\ f_{xx} &= 6x, \\ f_{yy} &= -6y, \\ f_{xy} &= 1, \\ f_{yx} &= 1. \end{aligned}$$

Thus the Hessian determinant is

$$\det(Hf) = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} 6x & 1 \\ 1 & -6y \end{pmatrix} = -36xy - 1.$$

Since $\det(Hf)(0,0) = -1 < 0$ we see that $(0,0)$ is a saddle point. Since $\det(Hf)(1/3,-1/3) = -36(-1/9) - 1 = 3 > 0$ we see that $(1/3,-1/3)$ is a local maximum or minimum. Since $f_{xx}(1/3,-1/3) = 6(1/3) = 2 > 0$ we see that $(1/3,-1/3)$ is a local minimum.[3]

[2]One can check that $\langle f_x(0,0), f_y(0,0)\rangle = \langle 0,0\rangle$ and $\langle f_x(1/3,-1/3), f_y(1/3,-1/3)\rangle = \langle 0,0\rangle$, as claimed.

[3]We could also check that $f_{yy}(1/3,-1/3) = -6(-1/3) = 2 > 0$.

Here is a picture: