Math 211 **Summer 2022**
Quiz 1 **Drew Armstrong**

Problem 1. Parametrized Paths. Consider the following parametrized path in $\mathbb{R}^2$:

$$f(t) = (x(t), y(t)) = (1 + 3t^2, 4t^2).$$

(a) Compute the velocity vector $f'(t)$ and the speed $\|f'(t)\|$.
(b) Compute the arc length between $t = 0$ and $t = 1$.

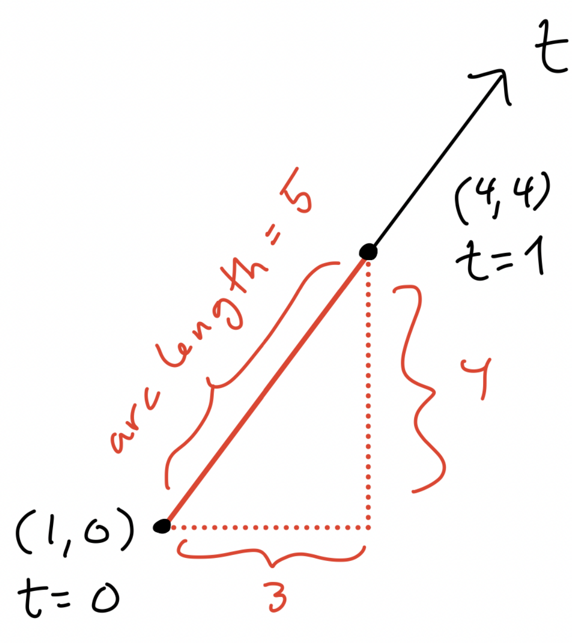
(a): The velocity is $f'(t) = (x'(t), y'(t)) = (4t, 8t)$ and the speed is

$$\begin{aligned} \|f'(t)\| &= \sqrt{(4t)^2 + (8t)^2} \\ &= \sqrt{16t^2 + 64t^2} \\ &= \sqrt{100t^2} \\ &= 10t. \end{aligned}$$

(b): The distance traveled between times $t = 0$ and $t = 1$ is

$$\begin{aligned} \int_0^1 \text{speed}\, dt &= \int_0^1 10t\, dt \\ &= 10 \left[t^2/2\right]_0^1 \\ &= 10\,[1/2 - 0/2] \\ &= 5. \end{aligned}$$

Remark: Actually this curve is a straight line with a non-standard parametrization. Instead of having constant speed, it gets faster as time increases. Knowing this, we could simply use the Pythagorean theorem to compute the arc length. Here is a picture:



Problem 2. Vector Arithmetic. Consider the triangle in $\mathbb{R}^3$ with vertices

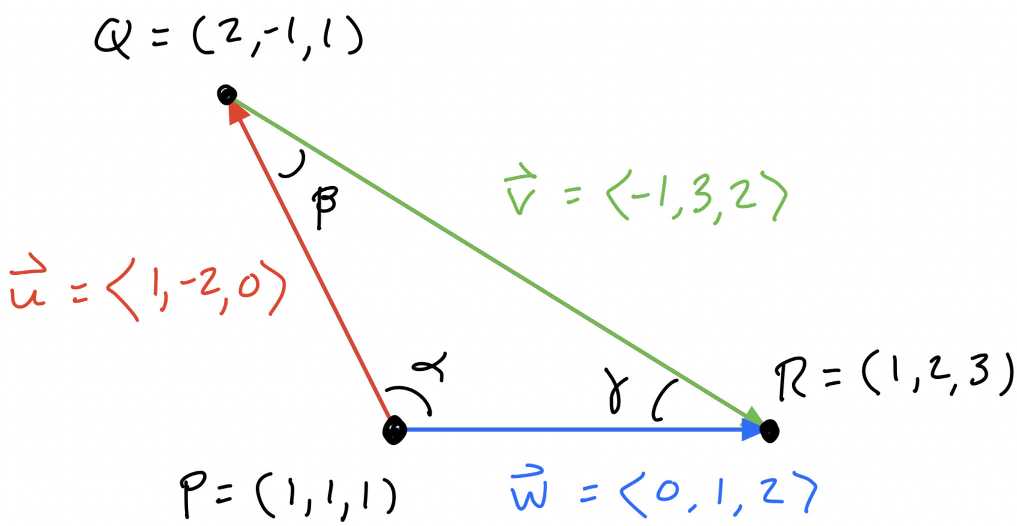
$$P = (1, 1, 1), \quad Q = (2, -1, 1), \quad R = (1, 2, 3).$$

(a) Compute the cosines of the angles of the triangle. [No need to find the actual angles.]
(b) Find the equation of the plane in $\mathbb{R}^3$ that contains this triangle.

(a): Consider the triangle with side vectors

$$\begin{aligned} \mathbf{u} &= \vec{PQ} = \langle 1, -2, 0 \rangle, \\ \mathbf{v} &= \vec{QR} = \langle -1, 3, 2 \rangle, \\ \mathbf{w} &= \vec{PR} = \langle 0, 1, 2 \rangle, \end{aligned}$$

and let α, β, γ be the angles as the points P, Q, R, respectively. Here is a picture:[1]



In order to compute the angles we first compute the dot products:

$$\mathbf{u} \bullet \mathbf{u} = 5, \quad \mathbf{u} \bullet \mathbf{v} = -7, \quad \mathbf{u} \bullet \mathbf{w} = -2, \quad \mathbf{v} \bullet \mathbf{v} = 14, \quad \mathbf{v} \bullet \mathbf{w} = 7, \quad \mathbf{w} \bullet \mathbf{w} = 5.$$

Then we have

$$\begin{aligned} \cos\alpha &= \frac{\mathbf{u} \bullet \mathbf{w}}{\sqrt{\mathbf{u} \bullet \mathbf{u}} \cdot \sqrt{\mathbf{w} \bullet \mathbf{w}}} = \frac{-2}{\sqrt{5} \cdot \sqrt{5}}, \\ \cos\beta &= \frac{-\mathbf{u} \bullet \mathbf{v}}{\sqrt{\mathbf{u} \bullet \mathbf{u}} \cdot \sqrt{\mathbf{v} \bullet \mathbf{v}}} = \frac{7}{\sqrt{5} \cdot \sqrt{14}}, \\ \cos\gamma &= \frac{\mathbf{v} \bullet \mathbf{w}}{\sqrt{\mathbf{v} \bullet \mathbf{v}} \cdot \sqrt{\mathbf{w} \bullet \mathbf{w}}} = \frac{7}{\sqrt{14} \cdot \sqrt{5}}. \end{aligned}$$

According to my computer this gives $\alpha = 113.6°$, $\beta = 33.2°$ and $\gamma = 33.2°$. Since these add to $180°$ it seems that I did not make a mistake.

[1] I drew this picture **after** I computed the angles, so it looks reasonably correct. Actually, **any** picture of the triangle is correct from some point of view. My picture is correct if we look at the triangle from directly above its plane.

(b): The equation of a plane is determined by one point (x_0, y_0, z_0) in the plane and a normal vector $\langle a, b, c\rangle$. To obtain a normal vector we can take the cross product of any two vectors in the plane. For example, let's take $\mathbf{u} = \langle 1, -2, 0\rangle$ and $\mathbf{w} = \langle 0, 1, 2\rangle$. Their cross product is

$$\begin{aligned}
\mathbf{u} \times \mathbf{w} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \det \begin{pmatrix} -2 & 0 \\ 1 & 2 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \mathbf{k} \\
&= -4\mathbf{i} - 2\mathbf{j} + 1\mathbf{k} \\
&= \langle -4, -2, 1\rangle.
\end{aligned}$$

Choosing the normal vector $\langle a, b, c\rangle = \mathbf{u} \times \mathbf{w} = \langle -4, -2, 1\rangle$ and the point $(x_0, y_0, z_0) = P = (1, 1, 1)$ gives the equation of the plane:

$$\begin{aligned}
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \\
-4(x - 1) - 2(y - 1) + 1(z - 1) &= 0 \\
-4x + 4 - 2y + 2 + z - 1 &= 0 \\
-4x - 2y + z &= -5 \\
4x + 2y - z &= 5.
\end{aligned}$$

Let's verify that each of the points P, Q, R is on this plane:

$$\begin{aligned}
4(1) + 2(1) - 1(1) &= 5, \\
4(2) + 2(-1) - 1(1) &= 5, \\
4(1) + 2(2) - 1(3) &= 5.
\end{aligned}$$

Yup.