Math 211 **Summer 2022**
Homework 4 **Drew Armstrong**

Problem 1. Integration over a Rectangle. Let $f(x,y) = 6x^2y$ and consider the rectangle R where $-1 \leq x \leq 1$ and $0 \leq y \leq 4$.

(a) Compute the integral $\iint_R f(x,y)\,dxdy$ by integrating over x first.
(b) Compute the integral $\iint_R f(x,y)\,dxdy$ by integrating over y first. Observe that you get the same answer.

Problem 2. Polar Coordinates. Cartesian coordinates (x,y) and polar coordinates (r,θ) are related as follows:

$$\left\{ \begin{array}{ll} x & = r\cos\theta \\ y & = r\sin\theta \end{array} \right\} \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} r & = \sqrt{x^2+y^2} \\ \theta & = \arctan(y/x) \end{array} \right\}$$

We will use the following notation[1] for the determinants of the Jacobian matrices:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} x_r & x_\theta \\ y_r & y_\theta \end{pmatrix} \quad \text{and} \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \det\begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}.$$

(a) Compute $\partial(x,y)/\partial(r,\theta)$.
(b) Compute $\partial(r,\theta)/\partial(x,y)$ and verify that

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

Problem 3. Integration Over a Tetrahedron. Let E be the solid tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$.

(a) Find a parametrization for this region.
(b) Use your parametrization to compute the volume of E.

Problem 4. Spherical Coordinates. Consider the solid region $E \subseteq \mathbb{R}^3$ that is inside the sphere $x^2+y^2+z^2 \leq 1$ and above the cone $z^2 = x^2+y^2$ with $z \geq 0$. Assume that this region has constant density 1 unit of mass per unit of volume.

(a) Use spherical coordinates to compute the mass $m = \iiint_E 1\,dV$.
(b) Compute the moment about the xy-plane, $M_{xy} = \iiint_E z\,dV$, and use this to find the center of mass. [Hint: Because the shape has rotational symmetry around the z-axis we know that $M_{xz} = M_{yz} = 0$.]

Problem 5. Volume of an Ellipsoid. Let a,b,c be positive.

(a) Use spherical coordinates to compute the volume of the unit sphere: $x^2+y^2+z^2 = 1$.
(b) Use the change of variables $(x,y,z) = (au,bv,cw)$ and part (a) to compute the volume of the ellipsoid: $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.

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[1] Warning: Just as dy/dx is not a quotient of numbers, $\partial(x,y)/\partial(r,\theta)$ is not a quotient of numbers. It's just a notation for the determinant of the Jacobian matrix.