Math 211 **Summer 2022**
Homework 3 **Drew Armstrong**

Problem 1. Tangent Line to an Ellipse. Let $a, b > 0$ and consider the ellilpse

$$ax^2 + by^2 = 1.$$

(a) Let $P = (x_0, y_0)$ be a point on the ellipse. Show that the tangent line at P has equation

$$ax_0x + by_0y = 1.$$

[Hint: Think of the ellipse as the level curve $f(x, y) = 1$ where $f(x, y) = ax^2 + by^2$.]

(b) Draw a picture of the ellipse and tangent line when $a = 1$, $b = 3$ and $P = (1/2, 1/2)$.

(a): Recall, for a general curve $f(x, y) =$ constant, the equation of the tangent plane at a point (x_0, y_0) is

$$\begin{aligned}
\nabla f(x_0, y_0) \bullet \langle x - x_0, y - y_0\rangle &= 0\\
\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \bullet \langle x - x_0, y - y_0\rangle &= 0\\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) &= 0.
\end{aligned}$$

In our case we can take $f(x, y) = ax^2 + by^2$, so that

$$\nabla f(x_0, y_0) = \langle 2ax_0, 2by_0\rangle.$$

Hence the tangent line has equation

$$(2ax_0)(x - x_0) + (2by_0)(y - y_0) = 0.$$

This can be simplified by using the fact that (x_0, y_0) is a point on the curve, i.e., that $ax_0^2 + by_0^2 = 1$. Then we have

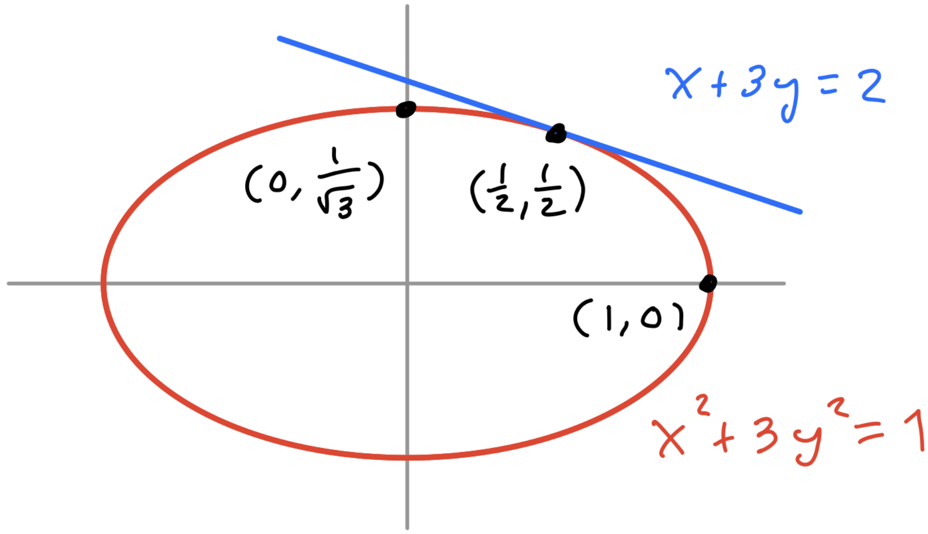
$$\begin{aligned}
(2ax_0)(x - x_0) + (2by_0)(y - y_0) &= 0\\
ax_0x + by_0y - (ax_0^2 + by_0^2) &= 0\\
ax_0x + by_0y - 1 &= 0\\
ax_0x + by_0y &= 1.
\end{aligned}$$

How nice.

(b): Now let $a = 1$, $b = 3$ and $(x_0, y_0) = (1/2, 1/2)$. The formula in part (a) tells us that the equation of the tangent line to the ellipse $x^2 + 3y^2 = 1$ at the point $(1/2, 1/2)$ is

$$\begin{aligned}
(1)(1/2)x + (3)(1/2)y &= 1\\
x + 3y &= 2.
\end{aligned}$$

Here is a picture:



Problem 2. Multivariable Chain Rule Practice. Let $f(x, y)$ be a function of x and y, where $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$ are functions of r and θ.

(a) Express f_r and f_θ in terms of r, θ, f_x and f_y.

(b) Express f_{rr} in terms of $r, \theta, f_{xx}, f_{yy}$ and f_{xy}. [Hint: Use the formulas $f_{xr} = f_{xx}\frac{dx}{dr} + f_{xy}\frac{dy}{dr} = f_{xx}\cos\theta + f_{xy}\sin\theta$ and $f_{yr} = f_{yx}\frac{dx}{dr} + f_{yy}\frac{dy}{dr} = f_{yx}\cos\theta + f_{yy}\sin\theta$.]

(a): Recall the multivariable chain rule. Since f is a function of x and y, we have for any other variable t that

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}.$$

In our case we are interested in $t = r$ and $t = \theta$. Taking $t = r$ gives

$$\begin{aligned}\frac{df}{dr} &= \frac{df}{dx} \cdot \frac{dx}{dr} + \frac{df}{dy} \cdot \frac{dy}{dr} \\ \frac{df}{dr} &= \frac{df}{dx} \cdot \cos\theta + \frac{df}{dy} \cdot \sin\theta \\ f_r &= f_x c\dot{o}s\theta + f_y \cdot \sin\theta,\end{aligned}$$

and taking $t = \theta$ gives

$$\begin{aligned}\frac{df}{d\theta} &= \frac{df}{dx} \cdot \frac{dx}{d\theta} + \frac{df}{dy} \cdot \frac{dy}{d\theta} \\ \frac{df}{d\theta} &= \frac{df}{dx} \cdot (-r\sin\theta) + \frac{df}{dy} \cdot r\cos\theta \\ f_\theta &= f_x \cdot (-r\sin\theta) + f_y \cdot r\cos\theta.\end{aligned}$$

Here we can think of $f_\theta(r, \theta), f_x(r, \theta), f_y(r, \theta)$ as some functions of r and θ, but we don't know what these functions are because we didn't give a formula for f.

(b): Our goal is to compute f_{rr}, which is the derivative of f_r with respect to r. There are many ways to write this:[1]

$$f_{rr} = (f_r)_r = \frac{d}{dr}\left(\frac{df}{dr}\right) = \frac{d^2 f}{dr^2}.$$

We assume that θ is not a function of r, so that[2]

$$\begin{aligned} f_{rr} &= (f_r)_r \\ &= (f_x \cdot \cos\theta + f_y \cdot \sin\theta)_r \\ &= f_{xr} \cdot \cos\theta + f_{yr} \cdot \sin\theta. \end{aligned}$$

Finally, we are asked to express f_{xr} an f_{yr} in terms of $r, \theta, f_{xx}, f_{yy}$ and f_{xy}. We do this by thinking of $f_x(x, y)$ as a function of x and y, and we use the multivariable chain rule:

$$\begin{aligned} f_{xr} &= \frac{df_x}{dr} \\ &= \frac{df_x}{dx} \cdot \frac{dx}{dr} + \frac{df_x}{dy} \cdot \frac{dy}{dr} \\ &= f_{xx} \cdot \cos\theta + f_{xy} \cdot \sin\theta. \end{aligned}$$

Similarly, we have

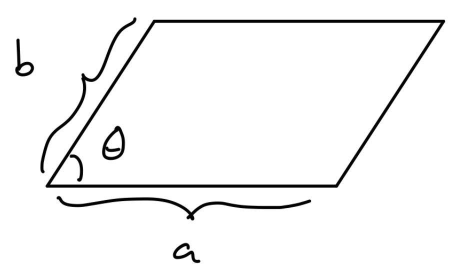
$$\begin{aligned} f_{yr} &= \frac{df_y}{dr} \\ &= \frac{df_y}{dx} \cdot \frac{dx}{dr} + \frac{df_y}{dy} \cdot \frac{dy}{dr} \\ &= f_{yx} \cdot \cos\theta + f_{yy} \cdot \sin\theta. \end{aligned}$$

Putting everything together gives

$$\begin{aligned} f_{rr} &= (f_{xx} \cdot \cos\theta + f_{xy} \cdot \sin\theta) \cdot \cos\theta + (f_{yx} \cdot \cos\theta + f_{yy} \cdot \sin\theta) \cdot \sin\theta \\ &= f_{xx} \cdot \cos^2\theta + f_{xy} \cdot (2\sin\theta\cos\theta) + f_{yy} \cdot \sin^2\theta. \end{aligned}$$

Wasn't that fun?

Problem 3. Linear Approximation. Consider a parallelogram with side lengths a, b and angle θ as follows:



(a) Find a formula for the area $A(a, b, \theta)$ in terms of a, b and θ.

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[1] And we can also replace the symbol "d" with ∂ to indicate that there are other variables floating around.

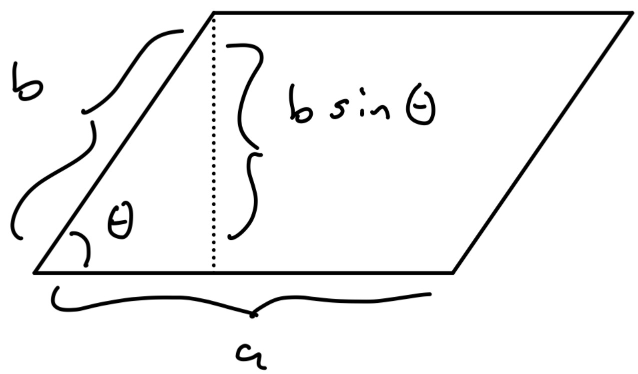
[2] If θ were a function of r we would need to use the product rule. For example, we would have $(f_x \cdot \cos\theta)_r = (f_x)_r \cdot \cos\theta + f_x \cdot (\cos\theta)_r = f_{xr} \cdot \cos\theta + f_x \cdot (-\sin\theta) \cdot \theta_r$. In our case we have $\theta_r = 0$.

(b) Suppose that we measure a, b, θ with the following uncertainties:

$$\begin{aligned} a &= 2 \pm 0.1 \text{ cm}, \\ b &= 1 \pm 0.1 \text{ cm}, \\ \theta &= 45 \pm 1 \text{ degrees}. \end{aligned}$$

Use these measurements together with your formula from part (a) to estimate the area.

(a): The area of a parallelogram is "base times height". In our case the height is $b\sin\theta$:



So the area is

$$A(a, b, \theta) = (\text{base})(\text{height}) = (a)(b\sin\theta) = ab\sin\theta.$$

(b): First we note that $A(a, b, \theta) \approx (2)(1)\sin(45^\circ) = \sqrt{2} = 1.414$. Next we will let dA, da, db and $d\theta$ denote tiny changes in these quantities. According to the chain rule, these infinitesimal quantities are related as follows:

$$\begin{aligned} dA &= \frac{dA}{da} \cdot da + \frac{dA}{db} \cdot db + \frac{dA}{d\theta} \cdot d\theta \\ &= b\sin\theta \cdot da + a\sin\theta \cdot db + ab\cos\theta \cdot d\theta. \end{aligned}$$

We are given that $a = 2$, $b = 1$, $\theta = 45^\circ$, $da = 0.1$, $db = 0.1$ and $d\theta = 1^\circ$. To keep the units consistent, the quantity $ab\cos\theta \cdot d\theta$ must have units of cm^2, which means that $d\theta$ must be measured in radians:[3]

$$d\theta = 1^\circ = \frac{1^\circ}{\cdot}\frac{2\pi}{360^\circ} = \frac{\pi}{180} = 0.01721420632.$$

Hence we obtain

$$\begin{aligned} dA &= (1)\sin(45^\circ)(0.1) + (2)\sin(45^\circ)(0.1) + (2)(1)\cos(45^\circ)(0.01721420632) \\ &= 0.2364765983. \end{aligned}$$

In summary, we have

$$A = 1.414 \pm 0.236 \text{ cm}^2.$$

Problem 4. Constrained Optimization. Let $f(x, y) = xy$ be a temperature distribution in the plane. Suppose that you travel around the unit circle $x^2 + y^2 = 1$ with parametrization $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ from $t = 0$ to $t = 2\pi$.

[3]Quote from a webpage: *For example, in the current SI, it is stated that angles are dimensionless based on the definition that an angle in radians is arc length divided by radius, so the unit is surmised to be a derived unit of one, or a dimensionless unit.*

(a) Let $T(t) = f(\mathbf{r}(t))$ be your temperature at time t. Compute $T'(t)$.
(b) Find all times t where $T(t)$ is maximized or minimized. [Hint: Set $T'(t) = 0$.]
(c) Use part (b) to find all points on the unit circle where the temperature is maximized or minimized.
(d) *Method of Lagrange Multipliers.* Now we express the circle as $g(x, y) = 1$ where $g(x, y) = x^2 + y^2$. Find all points on the circle where the vectors $\nabla g(x, y)$ and $\nabla f(x, y)$ point in the same direction. [Hint: Let $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some scalar λ. Use this and the equation $x^2 + y^2 = 1$ to solve for x and y. It's not as hard as it looks.]

(a): Our temperature at time t is

$$T(t) = f(\mathbf{r}(t)) = f(\cos t, \sin t) = (\cos t)(\sin t) = \frac{1}{2}\sin(2t),$$

hence our rate of change of temperature is

$$T'(t) = \frac{1}{2}\cos(2t) \cdot 2 = \cos(2t).$$

(b): From Calc I we know that local minima and maxima of $T(t)$ occur when $T'(t)$, i.e., when[4]

$$\begin{aligned} \cos(2t) &= 0 \\ 2t &= \pi/2 \text{ or } 3\pi/2 \\ t &= \pi/4 \text{ or } 3\pi/4 \text{ or } 5\pi/4 \text{ or } 7\pi/4. \end{aligned}$$

To tell whether these are maxima or minima we use the Calc I second derivative test. Since $T''(t) = 2\sin(2t)$, we have

$$\begin{aligned} T''(\pi/4) &= 2 > 0, \\ T''(3\pi/4) &= -2 < 0, \\ T''(5\pi/4) &= 2 > 0, \\ T''(7\pi/4) &= -2 < 0. \end{aligned}$$

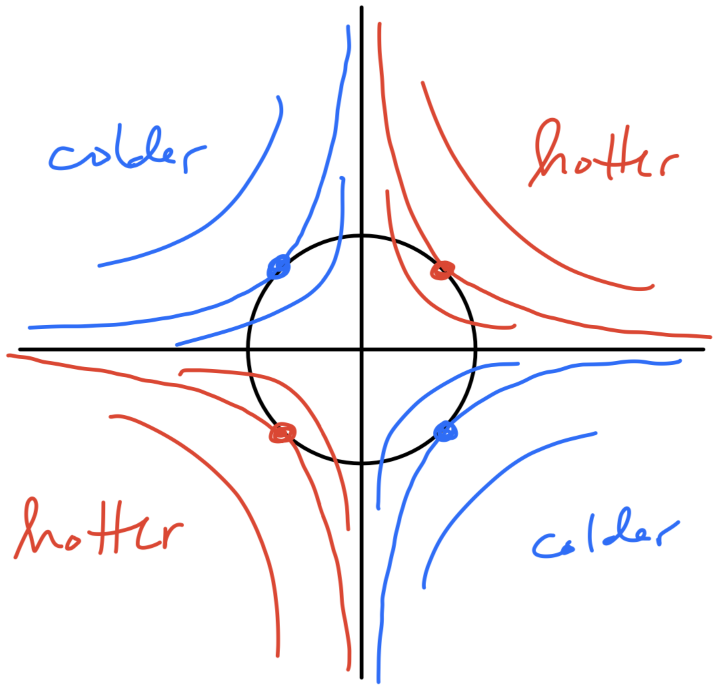
It follows that $t = \pi/4$ and $5\pi/4$ are local maxima, while $t = 3\pi/4$ and $7\pi/4$ are local minima.

(c): The corresponding points are

$$\begin{aligned} \mathbf{r}(\pi/4) &= \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle, \\ \mathbf{r}(3\pi/4) &= \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle, \\ \mathbf{r}(5\pi/4) &= \langle -1/\sqrt{2}, -1/\sqrt{2}\rangle, \\ \mathbf{r}(7\pi/4) &= \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle. \end{aligned}$$

Here is a picture:

[4] In general, given an angle θ and positive integer n, the expression θ/n refers to n different angles. For example, if $0 \le \theta < 2\pi$ then $\theta/2$ refers to the angles $\theta/2$ and $\theta/2 + \pi$.



(d): The method of Lagrange multipliers uses the fact (which is evident in the above picture) that extreme values occur when a level curve of $f(x, y)$ is tangent to the circle $x^2 + y^2 = 1$. Equivalently, if we define $g(x, y) = x^2 + y^2$ then a level curve of $f(x, y)$ is tangent to the curve $g(x, y) = 1$ when their gradient vectors are parallel:

$$\begin{aligned} \nabla f(x, y) &= \lambda g(x, y) \\ \langle y, x\rangle &= \lambda\langle 2x, 2y\rangle \\ \langle y, x\rangle &= \langle \lambda 2x, \lambda 2y\rangle. \end{aligned}$$

This gives two equations: $y = \lambda 2x$ and $x = \lambda 2y$. Solving for λ gives $\lambda = x/(2y) = y/(2x)$ and hence $2x^2 = 2y^2$, or $x^2 = y^2$. But we are only looking for points on the unit circle $x^2 + y^2 = 1$, hence we must have

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= 1/2 \\ x &= \pm 1/\sqrt{2}. \end{aligned}$$

This yields the same four points as in part (c), as it should.

Problem 5. Unconstrained Optimization. Let $f(x, y) = x^3 + 2xy - 4y^2 - 6x$ be a temperature distribution in the plane.

(a) Compute the gradient vector $\nabla f(x, y)$ and the Hessian determinant $\det(Hf)$.
(b) Find all critical points (x, y), i.e., all points where the gradient vector is zero:

$$\nabla f(x, y) = \langle 0, 0\rangle.$$

(c) Use the "second derivative test" to determine whether each critical point is a local maximum, local minimum, saddle point, or none of the above.

(a): We need to compute the derivatives f_x, f_y, f_{xx}, f_{yy} and f_{xy} (which equals f_{yx}):

$$\begin{aligned} f_x &= 3x^2 + 2y - 6, \\ f_y &= 2x - 8y, \\ f_{xx} &= 6x, \\ f_{yy} &= -8, \\ f_{xy} &= 2, \\ f_{yx} &= 2. \end{aligned}$$

Thus the gradient is

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 3x^2 + 2y - 6, 2x - 8y \rangle,$$

the Hessian matrix is

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & 2 \\ 2 & -8 \end{pmatrix},$$

and the Hessian determinant is

$$\det \begin{pmatrix} 6x & 2 \\ 2 & -8 \end{pmatrix} = -48x - 4.$$

(b): To find the critical points we set

$$\begin{aligned} \nabla f(x, y) &= \langle 0, 0 \rangle \\ \langle 3x^2 + 2y - 6, 2x - 8y \rangle &= \langle 0, 0 \rangle, \end{aligned}$$

which gives us two equations:

$$\begin{cases} 3x^2 + 2y - 6 &= 0, \\ 2x - 8y &= 0. \end{cases}$$

Solving the second equation for y gives $y = x/4$, then substituting into the first equation gives

$$\begin{aligned} 3x^2 + 2(x/4) - 6 &= 0 \\ 3x^2 + x/2 - 6 &= 0 \\ 6x^2 + x - 12 &= 0, \end{aligned}$$

which has solution

$$x = \frac{-1 \pm \sqrt{289}}{12} = \frac{-1 \pm 17}{12} = \frac{4}{3} \text{ and } -\frac{3}{2}.$$

The corresponding y values are

$$y = \frac{4/3}{4} = \frac{1}{3} \quad \text{and} \quad y = \frac{-3/2}{4} = -\frac{3}{8},$$

hence we have two critical points:

$$\left(\frac{4}{3}, \frac{1}{3}\right) \quad \text{and} \quad \left(-\frac{3}{2}, -\frac{3}{8}\right).$$

(c): To determine whether these are maxima, minima, saddle points, or none of the above, we use the Hessian determinant. For the point $(4/3, 1/3)$ we have

$$\det(Hf)(4/3, 1/3) = -48(4/3) - 4 = -68 < 0,$$

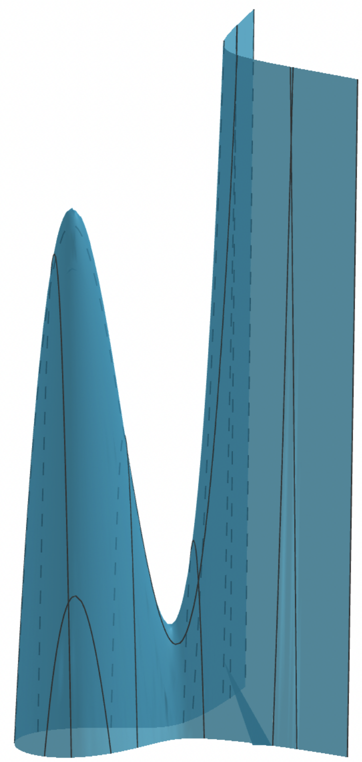
so this is a saddle point. For the point $(-3/2, -3/8)$ we have

$$\det(Hf)(-3/2, -3/8) = -48(-3/2) - 4 = 68 > 0,$$

so this is a local max or min. To determine which we can look at f_{xx} or f_{yy}.[5] Since

$$f_{xx}(-3/2, -3/8) = 6(-3/2) = -9 < 0,$$

we conclude that $(-3/2, -3/8)$ is a local maximum.[6] Thus our scalar field has one saddle point and one local maximum. Here is a picture of the graph:



[5]When $\det(Hf)(x_0, y_0) > 0$ then $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ are both nonzero and have the same sign. The textbook says to check the sign of f_{xx} but you can equally well check the sign of f_{yy}.

[6]In fact, since $f_{yy} = -8$ is constantly negative, this scalar field can have no local minima anywhere.