Math 211 **Summer 2022**
Homework 2 **Drew Armstrong**

Problem 1. A Line in Space. Consider the line in $\mathbb{R}^3$ passing through the two points

$$P = (-1, 2, 0) \quad \text{and} \quad Q = (3, 2, 1).$$

(a) Express this line in parametric form $\mathbf{r}(t) = (x_0 + ta, y_0 + tb, z_0 + tc)$.
(b) Find the equations of two planes in $\mathbb{R}^3$ whose intersection is this line. [Hint: There are infinitely many solutions. One solution uses the symmetric equations.]

(a): To express a line in parametric form we need one point on the line $P = (x_0, y_0, z_0)$ and one vector $\mathbf{v} = \langle a, b, c\rangle$ parallel to the line. Then we can write

$$\mathbf{r}(t) = P + t\mathbf{v} = (x_0 + ta, y_0 + tb, z_0 + tc).$$

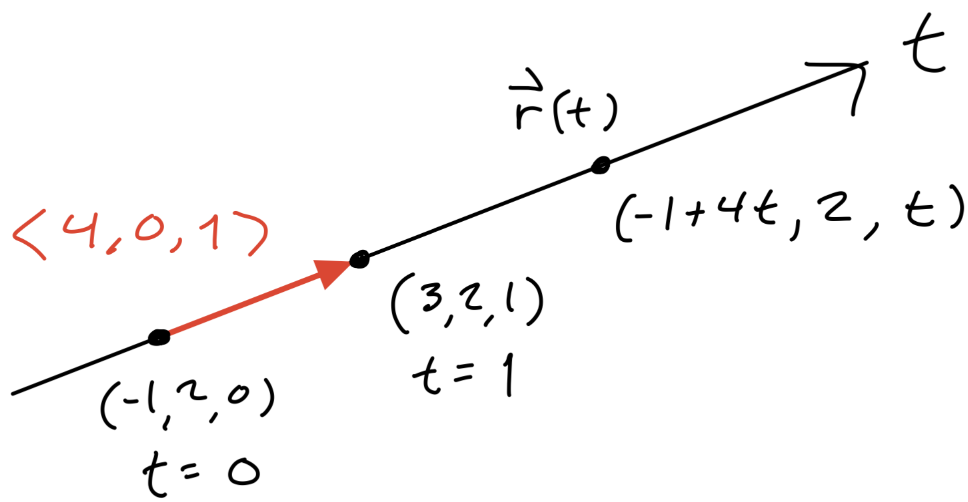
In our case we will take

$$P = (-1, 2, 0) \quad \text{and} \quad \vec{PQ} = \langle 3 - (-1), 2 - 2, 1 - 0\rangle = \langle 4, 0, 1\rangle,$$

so that

$$\mathbf{r}(t) = (-1 + 4t, 2 + 0t, 0 + 1t) = (-1 + 4t, 2, t).$$

Picture:



(b): We can eliminate t from the parametrization in part (a). To do this, we use the fact that $\mathbf{r}(t) = (x(t), y(t), z(t)) = (-1 + 4t, 2, t)$ to write

$$\left\{ \begin{array}{rcl} x &=& -1 + 4t \\ y &=& 2 \\ z &=& t \end{array} \right\} \implies \left\{ \begin{array}{rcl} t &=& (x+1)/4 \\ t &=& z \end{array} \right\}$$

The equation for y is unusual because it doesn't involve t. We observe that $y = 2$ is already the equation of a plane that contains both points $P = (-1, 2, 0)$ and $Q = (3, 2, 1)$. We can obtain another plane by equating the two expressions for t from above:

$$\begin{aligned}(x+1)/4 &= z\\ x+1 &= 4z\\ x-4z &= -1.\end{aligned}$$

The line from part (a) is the intersection of $y=2$ and $x-4z=-1$. Here is a picture:

More generally, any combination of these equations is another plane that contains the line from part (a). Namely, if we add k times the equation $x-4z=-1$ and ℓ times the equation $y=2$ then we obtain the plane

$$k(x-4z)+\ell y = k(-1)+2\ell.$$

For any values of k and ℓ this is the equation of a plane, and one can check that it contains the points $P=(-1,2,0)$ and $Q=(3,2,1)$.

Problem 2. An Intersection of Two Planes. Consider the following two planes in $\mathbb{R}^3$:

$$\begin{array}{c} (1) \\ (2) \end{array} \left\{ \begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1, \\ 2x & + & y & + & 3z & = & 0. \end{array} \right.$$

(a) Express the intersection of these planes as a parametrized line. [Hint: Multiply the first equation by 2 and then subtract the equations to obtain a new equation without x. Then let $t=z$ be a parameter and solve for x and y in terms of t.]

(b) We observe that $\mathbf{n}_1 = \langle 1,-1,2\rangle$ and $\mathbf{n}_2 = \langle 2,1,3\rangle$ are normal vectors for the two planes. Compute the cross product $\mathbf{n}_1 \times \mathbf{n}_2$. How is this vector related to the line in part (a)?

(a): First we multiply equation (1) by 2 and subtract this from equation (2) to obtain a new equation (3) that does not involve x:

$$(3) = (2) - 2(1) : 3y - z = -2.$$

Now we can use equation (3) to eliminate y from equation (1) or (2). Let's use (1). Multiplying (1) by 3 and adding (3) gives a new equation (4) that does not contain y:

$$(4) = 3(1) + (3) : 3x + 5z = 1.$$

Solving (3) and (4) for y and x, respectively, gives

$$\begin{aligned} x &= (1/3) - (5/3)z,\\ y &= -(2/3) + (1/3)z. \end{aligned}$$

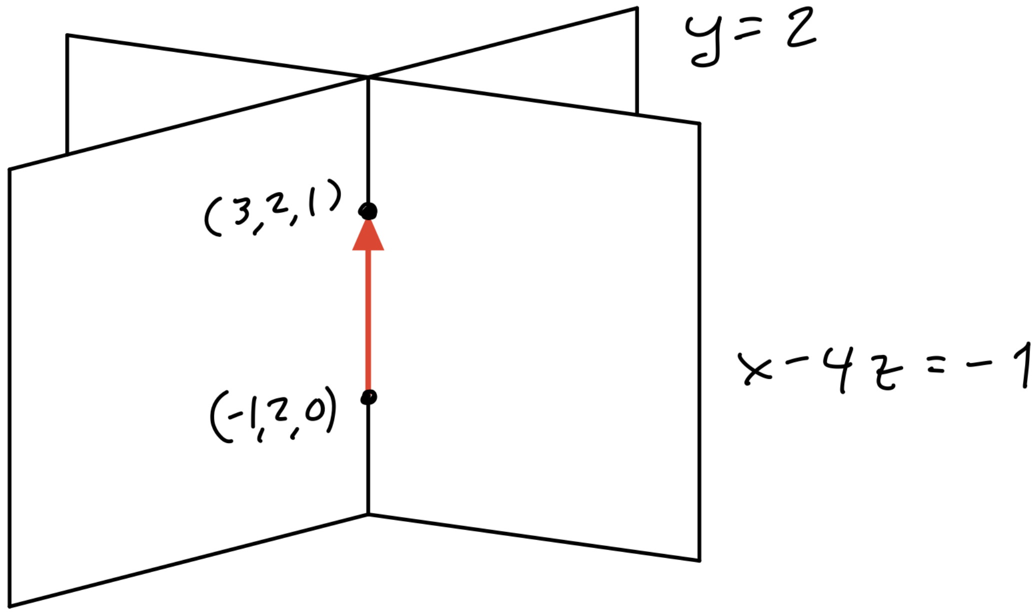
If we let $t=z$ be a parameter then we obtain a parametrized line:

$$\left\{ \begin{array}{ll} x & = (1/3) - (5/3)t, \\ y & = -(2/3) + (1/3)t, \\ z & = t, \end{array} \right.$$

which can also be expressed as

$$\begin{aligned} \mathbf{r}(t) &= \langle x(t), y(t), z(t)\rangle \\ &= \langle (1/3) - (5/3)t, -(2/3) + (1/3)t, t\rangle \\ &= \langle 1/3, -2/3, 0\rangle + t\langle -5/3, 1/3, 1\rangle. \end{aligned}$$

This is the line that begins at the point $P=(2/3,-2/3,0)$ and then travels with constant velocity $\mathbf{v} = \langle -5/3, 1/3, 1\rangle$. Here is a picture:



(b): On the other hand, let's consider the normal vectors of the planes (1) and (2), which are

$$\mathbf{n}_1 = \langle 1, -1, 2 \rangle \quad \text{and} \quad \mathbf{n}_2 = \langle 2, 1, 3 \rangle.$$

Their cross product is

$$\begin{aligned} \mathbf{n}_1 \times \mathbf{n}_2 &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \\ &= \det \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \mathbf{k} \\ &= (-3-2)\mathbf{i} - (3-4)\mathbf{j} + (1-(-2))\mathbf{k} \\ &= -5\mathbf{i} + \mathbf{j} + 3\mathbf{k} \\ &= \langle -5, 1, 3 \rangle. \end{aligned}$$

We observe that this is parallel to the velocity vector from part (a):

$$\langle -5, 1, 3 \rangle = 3\langle -5/3, 1/3, 1 \rangle.$$

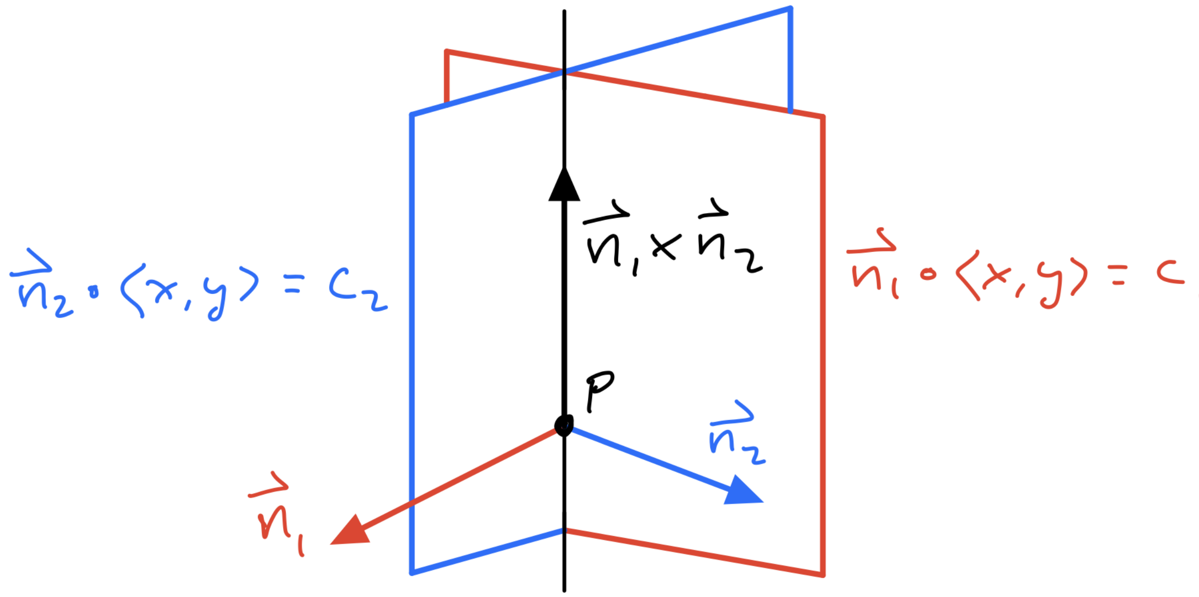
In fact, we could have used the cross product to solve (a) in a different way. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be any two vectors in $\mathbb{R}^3$, let c_1 and c_2 be constants, and consider the planes

$$\begin{cases} \mathbf{n}_1 \cdot \langle x, y \rangle &= c_1, \\ \mathbf{n}_2 \cdot \langle x, y \rangle &= c_2. \end{cases}$$

The intersection of these planes is a parametrized line of the form

$$\mathbf{r}(t) = P + t(\mathbf{n}_1 \times \mathbf{n}_2)$$

for some point P. Here is a picture (the red and blue vectors are perpendicular to the red and blue planes, respectively):



Problem 3. Projectile Motion. A projectile is launched from the point $(0,0)$ in $\mathbb{R}^2$ with an initial speed of 1, at an angle of θ above the horizontal. Thus we have

$$\mathbf{r}(0) = \langle 0, 0 \rangle,$$
$$\mathbf{r}'(0) = \langle \cos\theta, \sin\theta \rangle.$$

Let $g > 0$ be the constant of acceleration (which is 9.81 m/s^2 near the Earth).

(a) Use this information to compute the position $\mathbf{r}(t)$ at time t. [Hint: Neglecting air resistance, the acceleration due to gravity is constant: $\mathbf{r}''(t) = \langle 0, -g \rangle$.]
(b) When does the projectile hit the ground? Where does it land? [Hint: In part (a) you found formulas for $x(t)$ and $y(t)$ where $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Solve the equation $y(t) = 0$ for t. Your answer will involve the unknown constants θ and g.]
(c) Find the value of θ that **maximizes the horizontal distance traveled**. [Hint: The horizontal distance traveled is the x-coordinate $x(t)$ when the projectile lands, which you computed in part (b). Differentiate this distance with respect to θ.]

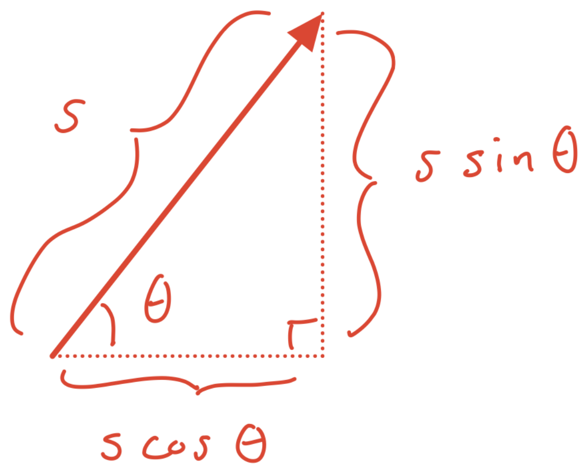
(a): I will solve a slightly more general problem. Consider a positive number $s > 0$ and let the initial velocity be $\mathbf{r}'(0) = \langle s\cos\theta, s\sin\theta \rangle$. Then the initial speed is[1]

$$\|\mathbf{r}'(0)\| = \sqrt{s^2\cos^2\theta + s^2\sin^2\theta} = \sqrt{s^2(\cos^2\theta + \sin^2\theta)} = \sqrt{s^2} = s.$$

That is, instead of specifying the initial velocity by its Cartesian coordinates, we will use the magnitude and direction. This idea is called "polar coordinates":

[1]The original statement of the problem had speed $s = 1$. Letting the speed be arbitrary doesn't really change anything, as we will see.

$$\vec{r}\,'(t) = \langle s\cos\theta, s\sin\theta \rangle$$



Our goal is to find explicit formulas for the position at time t. We begin by integrating $\mathbf{r}''(t) = \langle 0, -g \rangle$ to get $\mathbf{r}'(t)$. Since g is constant we have

$$\begin{aligned}\mathbf{r}'(t) &= \left\langle \int 0\,dt, \int -g\,dt \right\rangle \\ &= \langle c_1, -gt + c_2 \rangle\end{aligned}$$

for come constants of integration c_1, c_2. We use the initial velocity to see that

$$\langle s\cos\theta, s\sin\theta \rangle = \mathbf{r}'(0) = \langle c_1, 0 + c_2 \rangle = \langle c_1, c_2 \rangle,$$

and hence

$$\mathbf{r}'(t) = \langle s\cos\theta, -gt + s\sin\theta \rangle.$$

Now we integrate $\mathbf{r}'(t)$ to get $\mathbf{r}(t)$. Since s, θ and g are constant we have

$$\begin{aligned}\mathbf{r}(t) &= \left\langle \int s\cos\theta\,dt, \int (-gt + s\cos\theta)\,dt \right\rangle \\ &= \left\langle (s\cos\theta)t + c_3, -\frac{1}{2}gt^2 + (s\sin\theta)t + c_4 \right\rangle.\end{aligned}$$

We use the initial position to see that

$$\langle 0, 0 \rangle = \mathbf{r}(0) = \langle 0 + c_3, 0 + 0 + c_4 \rangle = \langle c_3, c_4 \rangle,$$

and hence

$$\mathbf{r}(t) = \left\langle (s\cos\theta)t, -\frac{1}{2}gt^2 + (s\sin\theta)t \right\rangle.$$

(b): We want to know when the projectile hits the ground. In other words, we want to solve

$$\begin{aligned} y(t) &= 0 \\ -\frac{1}{g}t^2 + (s\sin\theta)t &= 0 \\ t\left(-\frac{1}{2}gt + s\sin\theta\right) &= 0. \end{aligned}$$

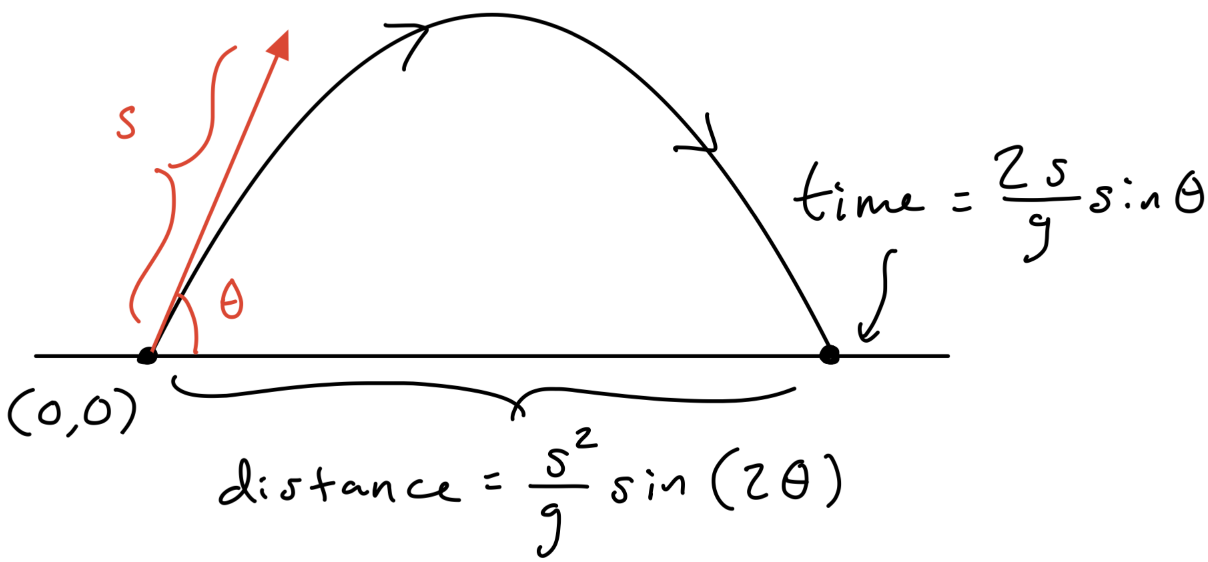
We find that the projectile is on the ground at time $t = 0$ (of course) and also when

$$\begin{aligned} -\frac{1}{2}gt + s\sin\theta = 0 \\ t = \frac{2s}{g}\sin\theta. \end{aligned}$$

(c): Now we want to know **where** the projectile hits the ground. Since it hits the ground at time $t = 2s\sin\theta/g$, the position when it hits the ground is[2]

$$\begin{aligned} \mathbf{r}\left(\frac{2s}{g}\sin\theta\right) &= \left\langle s\cos\theta\frac{2s}{g}\sin\theta, 0\right\rangle \\ &= \left\langle \frac{2s^2}{g}\sin\theta\cos\theta, 0\right\rangle \\ &= \left\langle \frac{s^2}{g}\sin(2\theta), 0\right\rangle. \end{aligned}$$

Here is a picture:



[2]Here I use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ to make the following computations simpler.

For which value of θ is the distance $s^2 \sin(2\theta)/g$ maximized? To solve this we will think of the distance as a function of θ, with s and g fixed:

$$f(\theta) = \frac{s^2}{g} \sin(2\theta).$$

Then to maximize $f(\theta)$ we take the derivative with respect to θ and set this equal to zero:

$$\begin{aligned} df/d\theta &= 0 \\ \frac{s^2}{g} \cos(2\theta) \cdot 2 &= 0 \\ \cos(2\theta) &= 0. \end{aligned}$$

We conclude that $2\theta = 90°$ and hence $\theta = 45°$. Summary: The horizontal distance of a cannonball is maximized by aiming the cannon at $45°$ above the horizontal. This is true on any planet and for any initial speed.

Problem 4. Some Vector Identities.

(a) Show that $\mathbf{v} \times \mathbf{v} = \langle 0, 0, 0 \rangle$ for any vector $\mathbf{v}$ in $\mathbb{R}^3$.
(b) Given any vector $\mathbf{r}$, we can define a *unit vector* $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ pointing in the same direction. Prove that $\|\mathbf{u}\| = 1$. [Hint: Use the formula $\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u} = (\mathbf{r}/\|\mathbf{r}\|) \bullet (\mathbf{r}/\|\mathbf{r}\|)$.]
(c) Let $\mathbf{r}(t)$ be a particle traveling on the surface of a sphere centered at $(0, 0, 0)$. In this case show that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all times t. [Hint: If c is the radius of the sphere then we have $\|\mathbf{r}(t)\|^2 = c^2$ for all times t. Rewrite this as $\mathbf{r}(t) \bullet \mathbf{r}(t) = c^2$ and differentiate both sides with respect to t. Use the product rule.]

(a): Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then we have

$$\mathbf{v} \times \mathbf{v} = \langle v_2 v_3 - v_3 v_2, v_3 v_1 - v_1 v_3, v_1 v_2 - v_2 v_1 \rangle = \langle 0, 0, 0 \rangle.$$

(b): Let $\mathbf{r}$ be any nonzero vector, so that $\|\mathbf{r}\| \neq 0$. Then we can define the vector $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$, i.e., $\mathbf{u} = (1/\|\mathbf{r}\|)\mathbf{r}$. To see that this vector has length 1 we use the dot product[3]

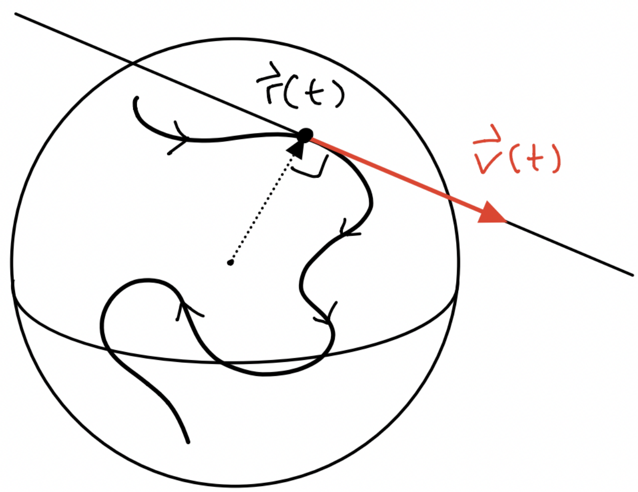
$$\begin{aligned} \|\mathbf{u}\|^2 &= \mathbf{u} \bullet \mathbf{u} \\ &= \left(\frac{1}{\|\mathbf{r}\|} \mathbf{r} \right) \bullet \left(\frac{1}{\|\mathbf{r}\|} \mathbf{r} \right) \\ &= \left(\frac{1}{\|\mathbf{r}\|} \right)^2 \mathbf{r} \bullet \mathbf{r} \\ &= \left(\frac{1}{\|\mathbf{r}\|} \right)^2 \|\mathbf{r}\|^2 \\ &= 1. \end{aligned}$$

[3]Recall that $(a\mathbf{u}) \bullet (b\mathbf{v}) = (ab)(\mathbf{u} \bullet \mathbf{v})$ for all constants a, b and vectors $\mathbf{u}, \mathbf{v}$.

(c): Let $\mathbf{c}$ be constant and consider a path $\mathbf{r}(t)$ such that $\|\mathbf{r}(t)\| = c$ for all times t. Then since the time derivative of c is equal to zero we get

$$\begin{aligned}
\|\mathbf{r}(t)\| &= c \\
\|\mathbf{r}(t)\|^2 &= c^2 \\
\mathbf{r}(t) \bullet \mathbf{r}(t) &= c^2 \\
[\mathbf{r}(t) \bullet \mathbf{r}(t)]' &= 0 \\
\mathbf{r}'(t) \bullet \mathbf{r}(t) + \mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\
2\mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0 \\
\mathbf{r}(t) \bullet \mathbf{r}'(t) &= 0.
\end{aligned}$$

This last equation tells us that the position vector $\mathbf{r}(t)$ is always perpendicular to the velocity vector $\mathbf{r}'(t)$. Interpretation: The velocity of a particle on the surface of a sphere is always tangent to the sphere. Picture:



Problem 5. Universal Gravitation. Choose a coordinate system with the sun at the origin $(0, 0, 0)$ in $\mathbb{R}^3$. According to Newton, a planet at position $\mathbf{r}(t)$ feels a gravitational force pointed directly toward the sun. The magnitude of this force is

$$\frac{GMm}{\|\mathbf{r}(t)\|^2},$$

where M is the mass of the sun, m is the mass of the planet and G is a constant of gravitation. For simplicity, let's assume that $G = M = m = 1$.

(a) Let $\mathbf{F}(t)$ be the gravitational force acting on the planet. Show that

$$\mathbf{F}(t) = -\mathbf{r}(t)/\|\mathbf{r}(t)\|^3.$$

[Hint: The vector $-\mathbf{r}(t)$ points from the planet to the sun, hence so does the vector $\mathbf{F}(t)$. Show that this $\mathbf{F}(t)$ has the correct magnitude.]

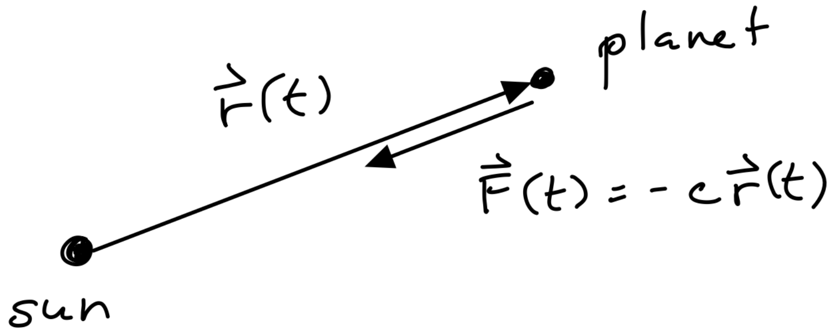
(b) Use part (b) to show that

$$\mathbf{r}''(t) = -\mathbf{r}(t)/\|\mathbf{r}(t)\|^3.$$

[Hint: Force equals mass times acceleration.]

(c) *Conservation of Angular Momentum.* Show that the vector $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, i.e., it does not depend on t. [Hint: Let $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t)$. Use the product rule and part (b) to show that $\mathbf{L}'(t) = \langle 0, 0, 0 \rangle$, hence $\mathbf{L}(t)$ is a constant vector.]

(a): Since the force $\mathbf{F}(t)$ points directly from the planet to the sun, we must have $\mathbf{F}(t) = -c\mathbf{r}(t)$ for some positive constant c. Here is a picture:



This implies that $\|\mathbf{F}(t)\| = c\|\mathbf{r}(t)\|$. On the other hand, Newton's law of universal gravitation tells us that $\|\mathbf{F}(t)\| = GMm/\|\mathbf{r}(t)\|^2 = 1/\|\mathbf{r}(t)\|^2$ (because we assume that $G = M = m = 1$). Putting these together gives

$$\begin{aligned} c\|\mathbf{r}(t)\| &= 1/\|\mathbf{r}(t)\|^2 \\ c &= 1/\|\mathbf{r}(t)\|^3, \end{aligned}$$

and hence

$$\mathbf{F}(t) = -c\mathbf{r}(t) = -\frac{1}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t).$$

(b): Newton's second law says that force equals mass times acceleration. In our case since $\mathbf{r}''(t)$ is the acceleration of the planet, $\mathbf{F}(t)$ is the force acting on the planet, and $m = 1$ is the mass of the planet, we have

$$\begin{aligned} m\mathbf{r}''(t) &= \mathbf{F}(t) \\ 1\mathbf{r}''(t) &= -\frac{1}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t) \\ \mathbf{r}''(t) &= -\frac{1}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t). \end{aligned}$$

This is a system of three coupled second order differential equations. To be explicit, suppose that $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. Then the above vector equation becomes three scalar equations:

$$\begin{cases} x''(t) &= -x(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ y''(t) &= -y(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}, \\ z''(t) &= -z(t)/[x'(t)^2 + y'(t)^2 + z'(t)^2]^{3/2}. \end{cases}$$

This is **much** more difficult to solve than the system in Problem 3.

(c): One of Newton's big accomplishments was to show that this system implies that planetary orbits are ellipses with the sun at one focus. You can find the details in last summer's course notes. This time we'll only discuss the first step, which is called *conservation of angular momentum*. The angular momentum at time t is defined as a cross product:[4]

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t).$$

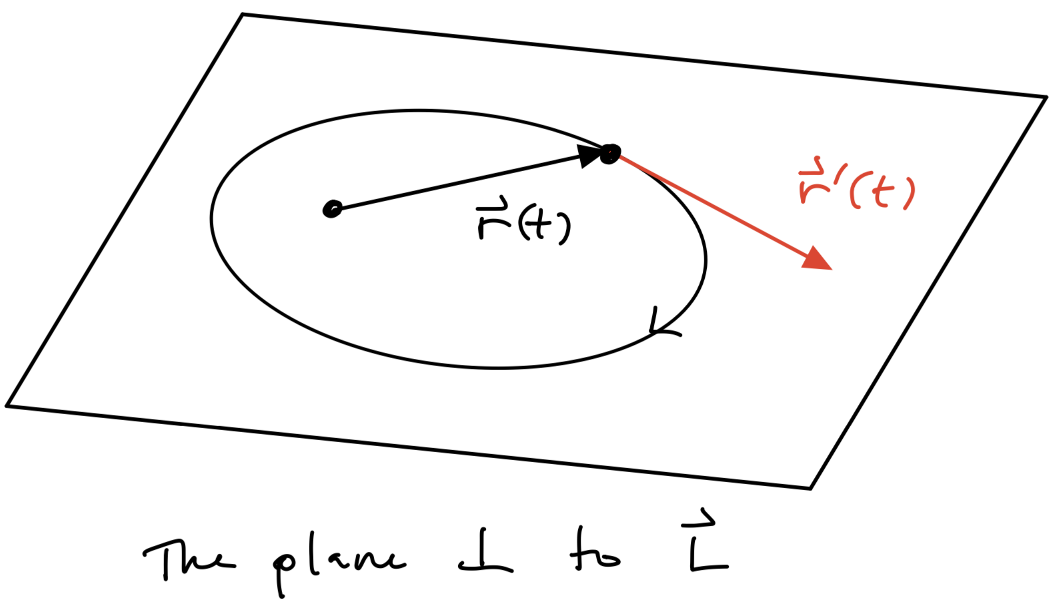
In principle the vector $\mathbf{L}(t)$ depends on time but we will show that it is actually constant. Using the product rule for differentiation of the cross product gives

$$\begin{aligned}\mathbf{L}'(t) &= [\mathbf{r}(t) \times \mathbf{r}'(t)]' \\ &= \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t).\end{aligned}$$

But $\mathbf{r}'(t) \times \mathbf{r}'(t) = \langle 0,0,0\rangle$ from Problem 4(a), and $\mathbf{r}''(t) = -\mathbf{r}(t)/\|\mathbf{r}(t)\|^3$ from part (b). Hence

$$\begin{aligned}\mathbf{L}'(t) &= \langle 0,0,0\rangle + \mathbf{r}(t) \times \left(-\frac{1}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t)\right) \\ &= \langle 0,0,0\rangle - \frac{1}{\|\mathbf{r}(t)\|^3}\mathbf{r}(t) \times \mathbf{r}(t) \\ &= \langle 0,0,0\rangle - \frac{1}{\|\mathbf{r}(t)\|^3}\langle 0,0,0\rangle \\ &= \langle 0,0,0\rangle.\end{aligned}$$

Hence $\mathbf{L} = \mathbf{L}(t)$ is constant. Since $\mathbf{L} = \mathbf{r}(t) \times \mathbf{r}'(t)$ it follows from this that the planet orbits in the plane perpendicular to the vector $\mathbf{L}$. We may choose coordinates so that the planet orbits in the x, y-plane and then the problem becomes two-dimensional. Here is a picture:



[4]It is actually $m\mathbf{r}(t) \times \mathbf{r}'(t)$ but we have assumed that $m = 1$.