Math 211 **Summer 2022**
Homework 2 **Drew Armstrong**

Problem 1. A Line in Space. Consider the line in $\mathbb{R}^3$ passing through the two points

$$P = (-1, 2, 0) \quad \text{and} \quad Q = (3, 2, 1).$$

(a) Express this line in parametric form $\mathbf{r}(t) = (x_0 + ta, y_0 + tb, z_0 + tc)$.
(b) Find the equations of two planes in $\mathbb{R}^3$ whose intersection is this line. [Hint: There are infinitely many solutions. One solution uses the symmetric equations.]

Problem 2. An Intersection of Two Planes. Consider the following two planes in $\mathbb{R}^3$:

$$\begin{cases} x - y + 2z = 1, \\ 2x + y + 3z = 0. \end{cases}$$

(a) Express the intersection of these planes as a parametrized line. [Hint: Multiply the first equation by 2 and then subtract the equations to obtain a new equation without x. Then let $t = z$ be a parameter and solve for x and y in terms of t.]
(b) We observe that $\mathbf{n}_1 = \langle 1, -1, 2\rangle$ and $\mathbf{n}_2 = \langle 2, 1, 3\rangle$ are normal vectors for the two planes. Compute the cross product $\mathbf{n}_1 \times \mathbf{n}_2$. How is this vector related to the line in part (a)?

Problem 3. Projectile Motion. A projectile is launched from the point $(0, 0)$ in $\mathbb{R}^2$ with an initial speed of 1, at an angle of θ above the horizontal. Thus we have

$$\mathbf{r}(0) = \langle 0, 0\rangle,$$
$$\mathbf{r}'(0) = \langle \cos\theta, \sin\theta\rangle.$$

Let $g > 0$ be the constant of acceleration (which is $9.81\ m/s^2$ near the Earth).

(a) Use this information to compute the position $\mathbf{r}(t)$ at time t. [Hint: Neglecting air resistance, the acceleration due to gravity is constant: $\mathbf{r}''(t) = \langle 0, -g\rangle$.]
(b) When does the projectile hit the ground? Where does it land? [Hint: In part (a) you found formulas for $x(t)$ and $y(t)$ where $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Solve the equation $y(t) = 0$ for t. Your answer will involve the unknown constants θ and g.]
(c) Find the value of θ that **maximizes the horizontal distance traveled**. [Hint: The horizontal distance traveled is the x-coordinate $x(t)$ when the projectile lands, which you computed in part (b). Differentiate this distance with respect to θ.]

Problem 4. Some Vector Identities.

(a) Show that $\mathbf{v} \times \mathbf{v} = \langle 0, 0, 0\rangle$ for any vector $\mathbf{v}$ in $\mathbb{R}^3$.
(b) Given any vector $\mathbf{r}$, we can define a *unit vector* $\mathbf{u} = \mathbf{r}/\|\mathbf{r}\|$ pointing in the same direction. Prove that $\|\mathbf{u}\| = 1$. [Hint: Use the formula $\|\mathbf{u}\|^2 = \mathbf{u} \bullet \mathbf{u} = (\mathbf{r}/\|\mathbf{r}\|) \bullet (\mathbf{r}/\|\mathbf{r}\|)$.]
(c) Let $\mathbf{r}(t)$ be a particle traveling on the surface of a sphere centered at $(0, 0, 0)$. In this case show that $\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0$ for all times t. [Hint: If c is the radius of the sphere then we have $\|\mathbf{r}(t)\|^2 = c^2$ for all times t. Rewrite this as $\mathbf{r}(t) \bullet \mathbf{r}(t) = c^2$ and differentiate both sides with respect to t. Use the product rule.]

Problem 5. Universal Gravitation. Choose a coordinate system with the sun at the origin $(0,0,0)$ in $\mathbb{R}^3$. According to Newton, a planet at position $\mathbf{r}(t)$ feels a gravitational force pointed directly toward the sun. The magnitude of this force is

$$\frac{GMm}{\|\mathbf{r}(t)\|^2},$$

where M is the mass of the sun, m is the mass of the planet and G is a constant of gravitation. For simplicity, let's assume that $G = M = m = 1$.

(a) Let $\mathbf{F}(t)$ be the gravitational force acting on the planet. Show that

$$\mathbf{F}(t) = -\mathbf{r}(t)/\|\mathbf{r}(t)\|^3.$$

[Hint: The vector $-\mathbf{r}(t)$ points from the planet to the sun, hence so does the vector $\mathbf{F}(t)$. Show that this $\mathbf{F}(t)$ has the correct magnitude.]

(b) Use part (b) to show that

$$\mathbf{r}''(t) = -\mathbf{r}(t)/\|\mathbf{r}(t)\|^3.$$

[Hint: Force equals mass times acceleration.]

(c) *Conservation of Angular Momentum.* Show that the vector $\mathbf{r}(t) \times \mathbf{r}'(t)$ is constant, i.e., it does not depend on t. [Hint: Let $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{r}'(t)$. Use the product rule and part (b) to show that $\mathbf{L}'(t) = \langle 0,0,0\rangle$, hence $\mathbf{L}(t)$ is a constant vector.]