Math 211 **Summer 2022**
Homework 1 **Drew Armstrong**

Problem 1. Lines and Circles. For each curve compute the velocity vector and the speed at time t. Also eliminate t to find an equation relating x and y.

(a) $(x, y) = (a + ut, b + vt)$ where a, b, u, v are constants.
(b) $(x, y) = (a + r\cos t, b + r\sin t)$ where a, b, r are constants.

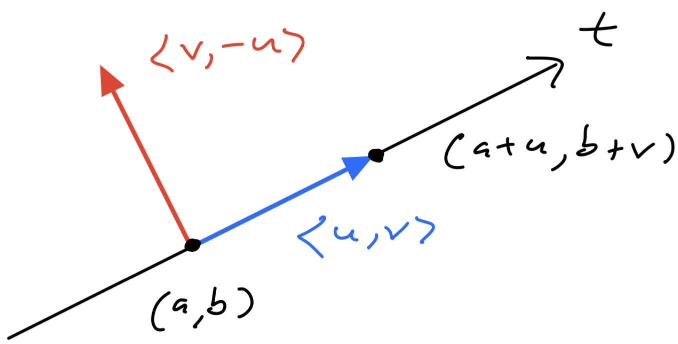
(a): The velocity and speed are

$$(dx/dt, dy/dt) = (u, v) \quad \text{and} \quad \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{u^2 + v^2}.$$

Note that these are both constant, i.e., they do not depend on t.[1] To eliminate t we will assume that $u \neq 0$ and $v \neq 0$, so that $x = a + ut$ implies $t = (x - a)/u$ and $y = b + vt$ implies $t = (y - b)/v$. Then equation these expressions for t gives

$$\begin{aligned} (x-a)/u &= (y-b)/v \\ v(x-a) &= u(y-b) \\ v(x-a) - u(y-b) &= 0. \end{aligned}$$

From our discussion in class we see that this line contains the point (a, b) and is perpendicular to the vector $\langle v, -u\rangle$. Here is a picture:



(b): The velocity and speed are

$$(dx/dt, dy/dt) = (-r\sin t, r\cos t)$$

and

$$\begin{aligned} \sqrt{(dx/dt)^2 + (dy/dt)^2} &= \sqrt{(-r\sin t)^2 + (r\cos t)^2} \\ &= \sqrt{r^2(\sin^2 t + \cos^2 t)} \\ &= \sqrt{r^2} \\ &= r. \end{aligned}$$

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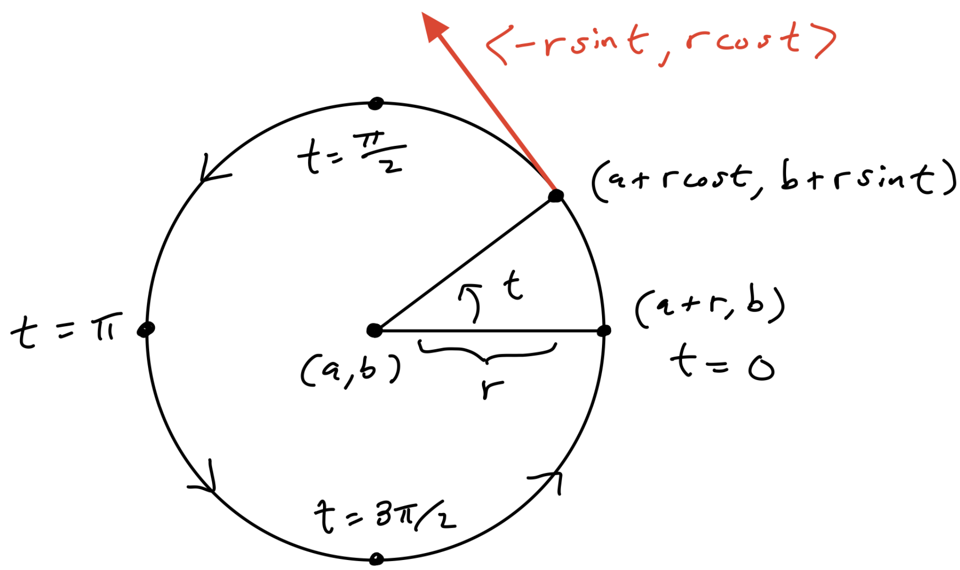
[1]Thus a parametrized line has constant velocity. Later we will see that any parametrized curve with constant velocity is a line.

We assume that r is positive, so $\sqrt{r^2} = |r| = r$. The speed is constant, but the velocity vector is not constant. We can eliminate t by using the trig identity $\sin^2 t + \cos^2 t = 1$ as follows:

$$(x-a)^2 + (y-b)^2 = (r\cos t)^2 + (r\sin t)^2 = r^2(\cos^2 t + \sin^2 t) = r^2.$$

This is the equation of a circle with radius r, centered at (a, b). Here is a picture:



Problem 2. An Interesting Curve. Consider the parametrized curve

$$(x, y) = (t^2 - 1, t^3 - t).$$

(a) Eliminate t to find an equation relating x and y. [Hint: Note that $y/x = t$.]
(b) Find the points on the curve where the tangent line is vertical, horizontal, or has slope $+1$ or -1. [Hint: The slope of the tangent at time t is $dy/dx = (dy/dt)/(dx/dt)$.]
(c) Use the information in part (b) to sketch the curve.

(a): Substitute $t = y/x$ into the equation $x = t^2 - 1$ to get

$$\begin{aligned} x &= (y/x)^2 - 1 \\ x &= y^2/x^2 - 1 \\ x^3 &= y^2 - x^2 \\ x^3 + x^2 &= y^2. \end{aligned}$$

(c): Let's write $f(t) = (t^2 - 1, t^3 - t)$. The velocity is $f'(t) = (dx/dt, dy/dt) = (2t, 3t^2 - 1)$, so the slope of the tangent line at time t is

$$\frac{dy}{dx} = \frac{dx/dt}{dy/dt} = \frac{3t^2 - 1}{2t}.$$

We note that this expression goes to ∞ as $t \to 0$, so there is a vertical tangent at the point $f(0) = (-1, 0)$. If the tangent line is horizontal (i.e., slope 0) then we have

$$\frac{3t^2 - 1}{2t} = 0 \quad \Rightarrow \quad 3t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm\sqrt{1/3}.$$

Hence the tangent line is horizontal at the points

$$f(-\sqrt{1/3}) \approx (-0.67, 0.4) \quad \text{and } f(\sqrt{1/3}) \approx (-0.67, -0.4).$$

If then tangent line has slope +1 then we have

$$\frac{3t^2 - 1}{2t} = 1 \quad \Rightarrow \quad 3t^2 - 1 = 2t \quad \Rightarrow \quad t = 1 \text{ or } -1/3,$$

which happens at the points

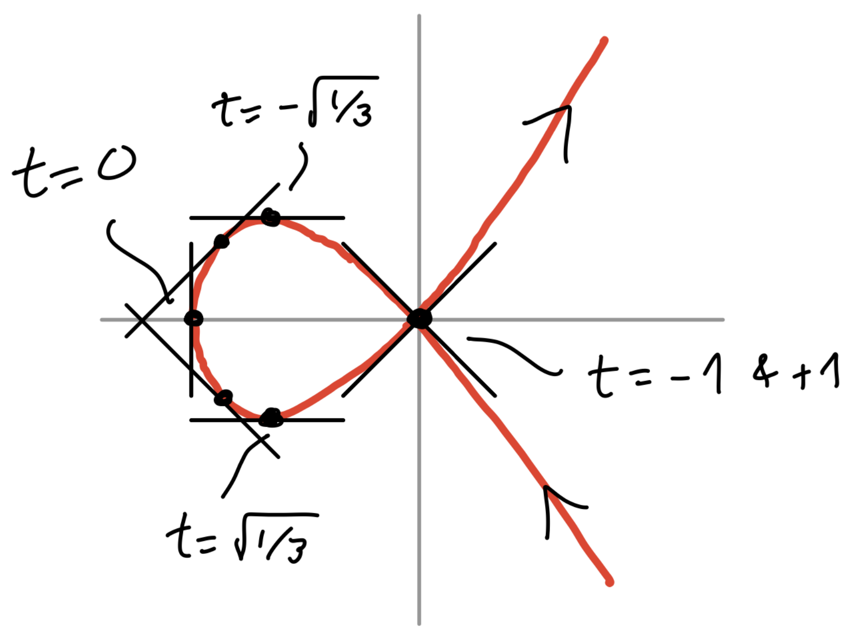
$$f(1) = (0,0) \quad \text{and} \quad f(-1/3) \approx (-0.89, 0.3)$$

A similar computation shows that the tangent line has slope −1 at the points

$$f(-1) = (0,0) \quad \text{and} \quad f(1/3) \approx (-0.89, -0.3).$$

Note that there are two different slopes at the point $(0,0)$ because the curve passes through this point twice, at times $t = -1$ and $t = +1$.

(c): Here is a picture:



Problem 3. The Cycloid. The cycloid is an interesting curve whose arc length can be computed by hand. It is parametrized by

$$(x, y) = (t - \sin t, 1 - \cos t).$$

(a) Sketch the curve between times $t = 0$ and $t = 2\pi$. [Hint: The slope of the tangent line at time t is $(dy/dt)/(dx/dt) = \sin t/(1 - \cos t)$, which goes to $+\infty$ as $t \to 0$ from the right and goes to $-\infty$ as $t \to 2\pi$ from the left. You do not need to prove this.]

(b) Compute the arc length between $t = 0$ and $t = 2\pi$. [Hint: You will need the trig identities $\sin^2 t + \cos^2 t = 1$ and $1 - \cos t = 2\sin^2(t/2)$.]

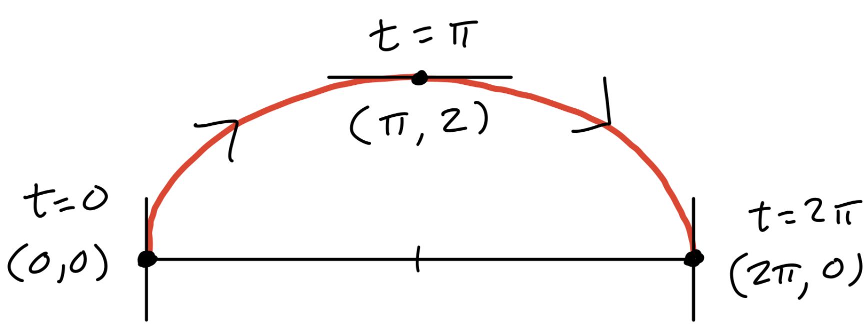
(a): Let $f(t) = (t - \sin t, 1 - \cos t)$, so the velocity is $f'(t) = (dx/dt, dy/dt) = (1 - \cos t, \sin t)$ and the slope of the tangent at time t is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$

The curve starts at the point $f(0) = (0, 0)$, where the tangent is vertical because $\sin t/(1 - \cos t) \to +\infty$ as $t \to 0$ (from the right). The curve ends at $f(2\pi) = (2\pi, 0)$, where the tangent is again vertical because $\sin t/(1 - \cos t) \to -\infty$ as $t \to 2\pi$ (from the left).[2] If the tangent is horizontal and $0 \leq t \leq 2\pi$ then

$$\frac{\sin t}{1 - \cos t} = 0 \quad \Rightarrow \quad \sin t = 0 \quad \Rightarrow \quad t = \pi,$$

which occurs at the point $f(\pi) = (\pi, 2)$. Here is a picture:



(b): We can use the trig identities $\sin^2 t + \cos^2 t = 1$ and $1 - \cos t = 2\sin^2(t/s)$ to simplify the speed of the parametrization as follows:

$$\begin{aligned}
\sqrt{(dx/dt)^2 + (dy/dt)^2} &= \sqrt{(1 - \cos t)^2 + (\sin t)^2} \\
&= \sqrt{1 - 2\cos t + \cos^2 + \sin^2 t} \\
&= \sqrt{1 - 2\cos t + 1} \\
&= \sqrt{2 - 2\cos t} \\
&= \sqrt{2(1 - \cos t)} \\
&= \sqrt{2 \cdot 2\sin^2(t/2)} \\
&= 2\sin(t/2),
\end{aligned}$$

which is non-negative because $0 \leq t \leq 2\pi$. Then the arc length between $t = 0$ and $t = 2\pi$ is the integral of the speed;

$$\begin{aligned}
\int_{t=0}^{t=2\pi} 2\sin(t/2)\, dt &= \int_{u=0}^{u=\pi} 2\sin u \cdot 2du \qquad [u = t/2, dt = 2du] \\
&= 4 \cdot [-\cos u]_{u=0}^{u=\pi} \\
&= 4 \cdot [-(-1) - (-1)] \\
&= 8.
\end{aligned}$$

Remarks:

[2] You do not need to prove this. The limits can be computed with L'Hopital's rule.

- It is possible to eliminate t as follows. First we rewrite $y = 1 - \cos t$ as

$$\begin{aligned} \cos t &= 1 - y \\ \cos^2 t &= 1 - 2y + y^2 \\ 1 - \cos^2 t &= 2y - y^2 \\ \sin^2 t &= y(2-y) \\ \sin t &= \sqrt{y(2-y)} \\ t &= \sin^{-1}\left(\sqrt{y(2-y)}\right). \end{aligned}$$

  Then we substitute these expressions for t and $\sin t$ into the expression for x to get

$$x = t - \sin t = \sin^{-1}\left(\sqrt{y(2-y)}\right) - \sqrt{y(2-y)}.$$

  What a mess. Clearly it is better to express the cycloid in terms of a parametrization.

- The cycloid is the answer to several interesting problems in physics. For example, suppose you have a pebble stuck in the surface of your car tire. As the car moves the pebble will follow a cycloidal path. Suppose that the tire has radius 1 unit, so the circumference is 2π units. As your travels a straight line distance of 2π units, the pebble will travel an arc length of 8 units.

Problem 4. A Triangle in Space. Consider the following points in $\mathbb{R}^3$:

$$P = (1, 1, -1), \quad Q = (1, -1, 1), \quad R = (-1, 1, 1).$$

(a) Find the coordinates of the three side vectors $\mathbf{u} = \vec{PQ}, \mathbf{v} = \vec{QR}, \mathbf{w} = \vec{PR}$.
(b) Use the length formula to compute the three side lengths $\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$.
(c) Use the dot product to compute the three angles of the triangle.

(a): Using the formula "head minus tail" gives

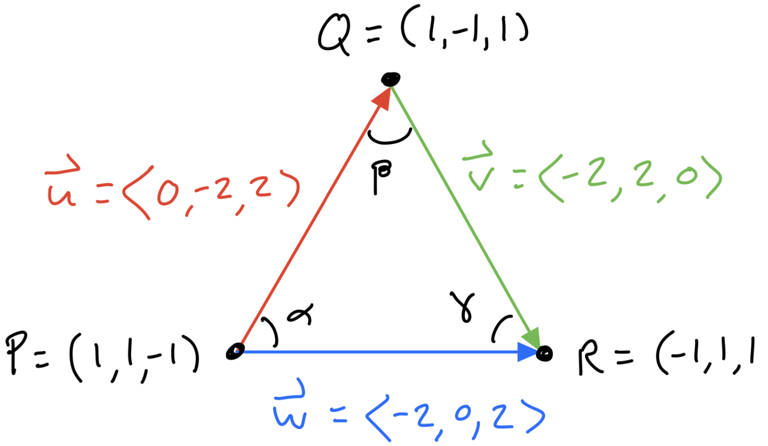
$$\begin{aligned} \mathbf{u} &= \vec{PQ} = \langle 1-1, -1-1, 1-(-1)\rangle = \langle 0, -2, 2\rangle, \\ \mathbf{v} &= \vec{QR} = \langle -1-1, 1-(-1), 1-1\rangle = \langle -2, 2, 0\rangle, \\ \mathbf{w} &= \vec{PR} = \langle -1-1, 1-1, 1-(-1)\rangle = \langle -2, 0, 2\rangle. \end{aligned}$$

(b): Using the formula for length gives

$$\begin{aligned} \|\mathbf{u}\| &= \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{(0)^2 + (-2)^2 + (2)^2} = \sqrt{8}, \\ \|\mathbf{v}\| &= \sqrt{\mathbf{v} \bullet \mathbf{v}} = \sqrt{(-2)^2 + (2)^2 + (0)^2} = \sqrt{8}, \\ \|\mathbf{w}\| &= \sqrt{\mathbf{w} \bullet \mathbf{w}} = \sqrt{(-2)^2 + (0)^2 + (2)^2} = \sqrt{8}. \end{aligned}$$

We see from this that the side lengths are equal, i.e., the triangle is equilateral. This implies that all three angles are $60°$, but we will check it anyway.

(c): Consider the picture



First we compute the dot products:

$$\mathbf{u} \bullet \mathbf{v} = (0)(-2) + (-2)(2) + (2)(0) = -4,$$
$$\mathbf{u} \bullet \mathbf{w} = (0)(-2) + (-2)(0) + (2)(2) = 4,$$
$$\mathbf{v} \bullet \mathbf{w} = (-2)(-2) + (2)(0) + (0)(2) = 4.$$

Since α is the angle between $\mathbf{u}$ and $\mathbf{w}$ we have

$$\cos\alpha = \frac{\mathbf{u} \bullet \mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|} = \frac{4}{\sqrt{8}\sqrt{8}} = \frac{1}{2}.$$

Since β is the angle between $-\mathbf{u}$ and $\mathbf{v}$ we have[3]

$$\cos\beta = \frac{(-\mathbf{u}) \bullet \mathbf{v}}{\|-\mathbf{u}\|\|\mathbf{v}\|} = \frac{-(\mathbf{u} \bullet \mathbf{v})}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{4}{\sqrt{8}\sqrt{8}} = \frac{1}{2}.$$

Since γ is the angle between $-\mathbf{v}$ and $-\mathbf{w}$ we have

$$\cos\gamma = \frac{(-\mathbf{v}) \bullet (-\mathbf{w})}{\|-\mathbf{v}\|\|-\mathbf{w}\|} = \frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{4}{\sqrt{8}\sqrt{8}} = \frac{1}{2}.$$

In any case, the angle is

$$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \text{ or } 300^\circ.$$

Remarks:

- Any two vectors placed tail to tail actually have two different angles between them, which have the same cosine. By convention we choose the smaller of these two angles.

- If we add a fourth point $S = (-1,-1,-1)$ then one can check that each of the triangles PQR, PQS, PRS and QRS is equilateral. Hence the four points $PQRS$ are the vertices of a regular tetrahedron in space. Furthermore, the center of the tetrahedron is at the origin $O = (0,0,0)$. The angle between any two vertices, measured at the origin, is called the *tetrahedral angle* θ. We can compute it as follows. Choose two random vertices, say P and Q and consider the vectors with tail at the origin:

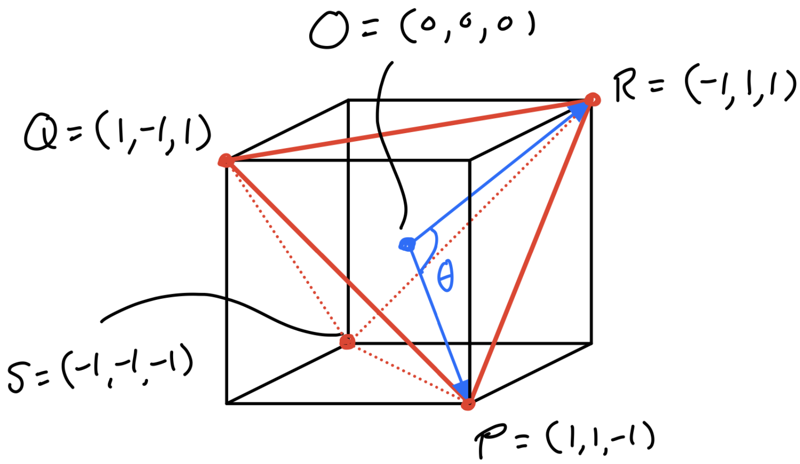
$$\vec{OP} = \langle 1, 1, -1\rangle,$$
$$\vec{OQ} = \langle 1, -1, 1\rangle.$$

[3]Recall: We measure the angle between vectors "tail to tail".

The tetrahedral angle satisfies

$$\cos\theta = \frac{\vec{OP} \bullet \vec{OQ}}{\|\vec{OP}\|\|\vec{OQ}\|} = \frac{-1}{\sqrt{3}\cdot\sqrt{3}} = -\frac{1}{3},$$

hence $\theta = \cos^{-1}(-1/3) \approx 109.5°$. Here is a picture:



Problem 5. Some Vector Arithmetic. Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors, living in 527-dimensional space. Use the rules of vector arithmetic (pages 112 and 147) to show that

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\bullet\mathbf{v}).$$

[Hint: Start with $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v})$. Now use FOIL and simplify the result.]

For any four vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$ we can use the distributive rule for the dot product to get

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\bullet(\mathbf{c}+\mathbf{d}) &= \mathbf{a}\bullet(\mathbf{c}+\mathbf{d}) + \mathbf{b}\bullet(\mathbf{c}+\mathbf{d}) \\ &= \mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{d} + \mathbf{b}\bullet\mathbf{c} + \mathbf{b}\bullet\mathbf{d}.\end{aligned}$$

This is a dot product version of FOIL. In our particular case we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\bullet\mathbf{u} - \mathbf{u}\bullet\mathbf{v} - \mathbf{v}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v}.\end{aligned}$$

Now we use the facts $\mathbf{u}\bullet\mathbf{u} = \|\mathbf{u}\|^2$, $\mathbf{v}\bullet\mathbf{v} = \|\mathbf{v}\|^2$ and $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$ to get

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\|^2 &= \mathbf{u}\bullet\mathbf{u} - \mathbf{u}\bullet\mathbf{v} - \mathbf{v}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v} \\ &= \mathbf{u}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v} - 2(\mathbf{u}\bullet\mathbf{v}) \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(\mathbf{u}\bullet\mathbf{v}).\end{aligned}$$

We discussed in class how this algebraic identity, together with the geometric Law of Cosines, leads to the theorem of the dot product:

$$\mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.$$

Problem 6. Equations of Lines and Planes. The equation of the line in $\mathbb{R}^2$ that contains the point (x_0, y_0) and is perpendicular to the vector $\mathbf{n} = \langle a, b\rangle$ is

$$a(x - x_0) + b(y - y_0) = 0.$$

The equation of the plane in $\mathbb{R}^3$ that contains the point (x_0, y_0, z_0) and is perpendicular to the vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

(a) Find the equation of the line containing $(2, 0)$ and perpendicular to $\langle 4, 3\rangle$.
(b) Find the equation of the plane containing $(1, 0, 0)$ and perpendicular to $\langle 1, 1, 1\rangle$.

(a): Not much to do here. The equation is

$$\begin{aligned} 4(x-2) + 3(y-0) &= 0 \\ 4x + 3y &= 8. \end{aligned}$$

(b): Again, not much to do. The equation is

$$\begin{aligned} 1(x-1) + 1(y-0) + 1(z-0) &= 0 \\ x + y + z &= 1. \end{aligned}$$

Why did I make this so easy? I guess I wasn't sure how far we would get in Thursday's lecture.

Remark: This plane contains the points P, Q, R from Problem 4. I probably should have rephrased Problem 5(b) to ask for the equation of the plane containing these points. In that case, we could get a normal vector by taking the cross product of any two vectors in the plane, say $\mathbf{u} = \vec{PQ}$ and $\mathbf{w} = \vec{PR}$:

$$\mathbf{n} = \mathbf{u} \times \mathbf{w} = \langle -2, 2, 0\rangle \times \langle -2, 0, 2\rangle.$$

Using the mnemonic gives

$$\begin{aligned} \mathbf{n} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix} \\ &= \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} -2 & 0 \\ -2 & 2 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} -2 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{k} \\ &= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \\ &= \langle 4, 4, 4\rangle. \end{aligned}$$

Then picking any point in the plane, say $P = (1, 1, -1)$, gives the equation

$$\begin{aligned} 4(x-1) + 4(y-1) + 4(z+1) &= 0 \\ 4x + 4y + 4z &= 4 \\ x + y + z &= 1. \end{aligned}$$

Yeah, that would have been a better problem.