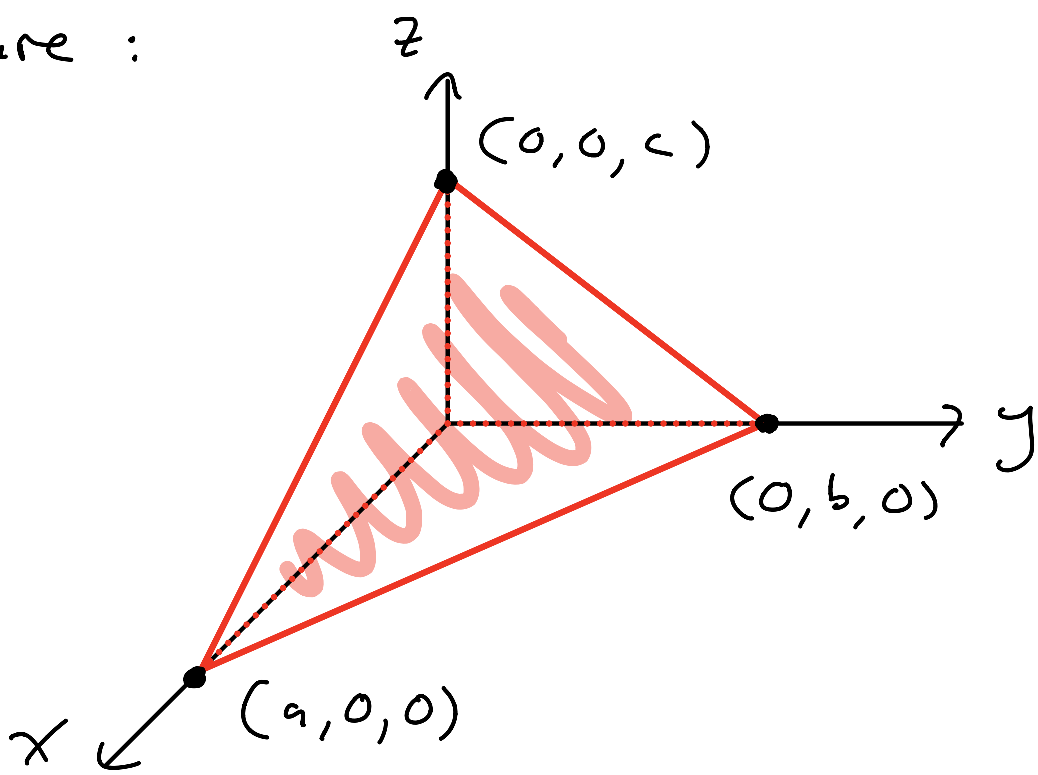
This note will provide hints for Homework 4, Problem 3.

"Integrating over a Tetrahedron":

Consider the tetrahedron in $\mathbb{R}^3$ with

$$x, y, z \geq 0 \quad \& \quad bcx + acy + abz \leq abc$$

Picture:



There are 6 basic ways to parametrize the tetrahedron.

Here is one way:

- Fix some x between 0 & a

Then a point $(x, y, 0)$ in the the tetrahedron must have

$x, y \geq 0$ & $bcx + acy \leq abc$

so that

$$0 \leq y \leq (abc - bcx)/ac$$

$$0 \leq y \leq b - bx/a$$

- For any (x, y, z) in the tetrahedron with (x, y) fixed, we must have

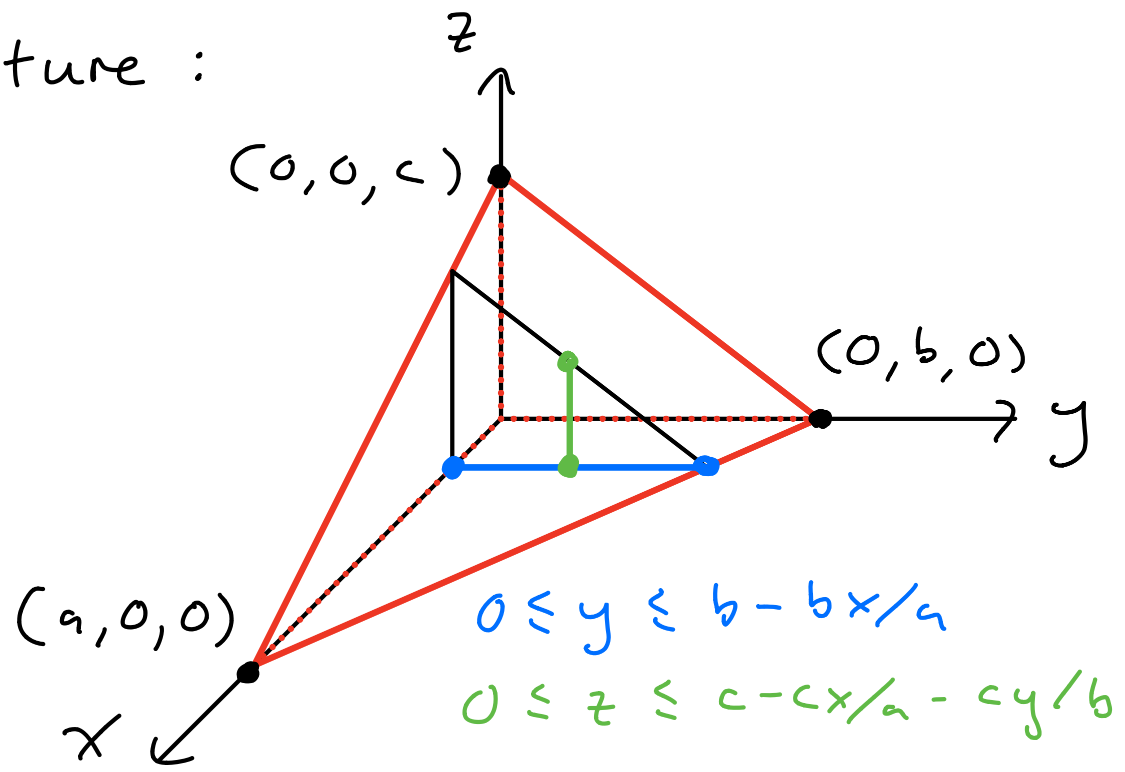
$z \geq 0$ & $bcx + acy + abz \leq abc$

so that

$$0 \leq z \leq (abc - bcx - acy)/ab$$

$$0 \leq z \leq c - cx/a - cy/b.$$

Picture :



The volume of the tetrahedron is

$$\text{vol} = \int_0^a \int_0^{b-\frac{b}{a}x} \int_0^{c-\frac{c}{a}x-\frac{c}{b}y} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b-\frac{b}{a}x} \left(c - \frac{c}{a}x - \frac{c}{b}y\right) dy\,dx$$

$$= \int_0^a \left(c - \frac{c}{a}x\right)\left(b - \frac{b}{a}x\right) - \frac{c}{2b}\left(b - \frac{b}{a}x\right)^2 dx$$

... computer

$$\text{vol} = abc/6$$

We can also compute the center of mass:

$$M_{yz} = \iiint x\,dV = \cdots = a^2bc/24$$

$$M_{xz} = \iiint y\,dV = \cdots = ab^2c/24$$

$$M_{xy} = \iiint z\,dV = \cdots = abc^2/24$$

so that

$$(\bar{x}, \bar{y}, \bar{z}) = (a/4, b/4, c/4).$$

Hmm... That is so nice that there must be an easier method.

However, if we work with a non-uniform density $\rho(x,y,z)$ then there is no shortcut!