HW 1 due before Friday's lecture.

Quiz 1 beginning of Monday's lecture.

Review of § 2.1, 2.2 :

A vector $\vec{v}$ is a directed line segment $\overrightarrow{PQ}$

$$P \xrightarrow{\vec{v}} Q$$

where point P is the tail of $\vec{v}$ & point Q is the head of $\vec{v}$.

In coordinates :

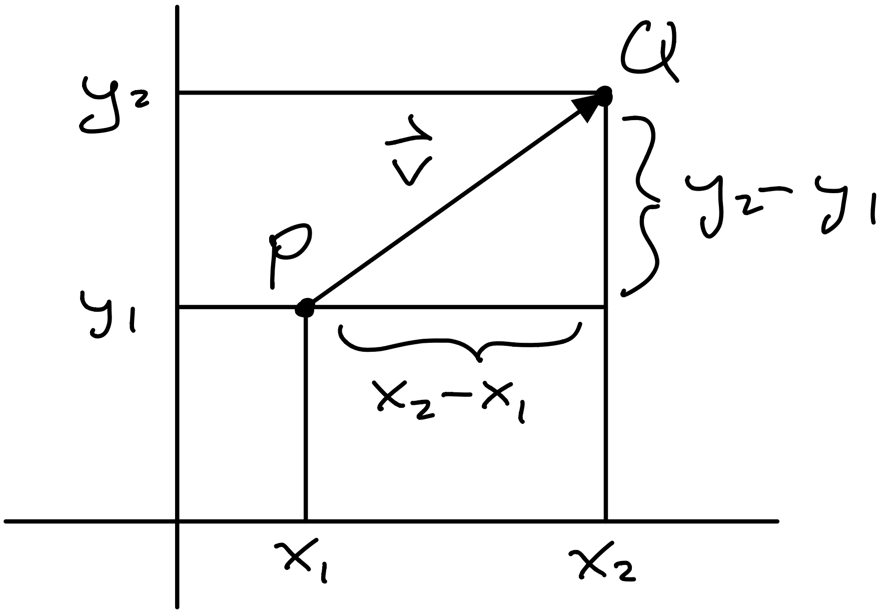
If $P = (x_1, y_1)$ & $Q = (x_2, y_2)$ then we say that

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

$$= \langle x_2, y_2 \rangle - \langle x_1, y_1 \rangle$$

= "head minus tail"

Picture :



The length $\|\vec{v}\|$ of the vector (also the distance between points P & Q is given by the Pyth. Thm. :

$$\|\vec{v}\|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\|\vec{v}\| = +\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

Sometimes we don't care about the endpoints and we just write

$$\vec{v} = \langle v_1, v_2 \rangle$$

$$\|\vec{v}\| = +\sqrt{v_1^2 + v_2^2}$$

In fact, it is easy to generalize this notation to "n-dimensional space" $\mathbb{R}^n$.

Definition of Vector Arithmetic.

Given two vectors

$$\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$$

$$\vec{v} = \langle v_1, v_2, \cdots, v_n \rangle$$

and a scalar k

["scalar" = "number"]

we define the sum & scalar product

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n \rangle,$$

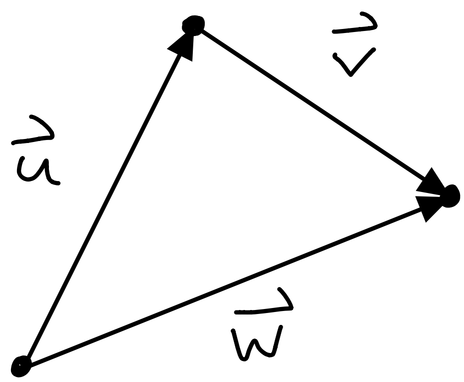
$$k\vec{v} = \langle kv_1, kv_2, \ldots, kv_n \rangle.$$

From these two operations we also get subtraction:

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$$
$$= \langle u_1 - v_1, u_2 - v_2, \dots, u_n - v_n \rangle.$$

Geometric Meaning:

- Addition & Subtraction



In this picture we have

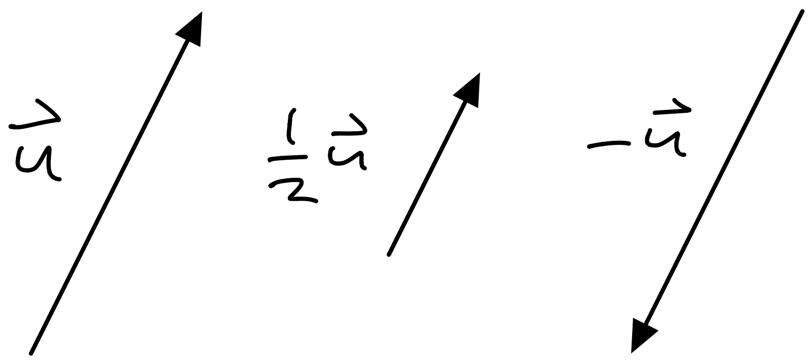
$$\vec{u} + \vec{v} = \vec{w}$$

and by adding $(-1)\vec{u}$ to both sides we also have

$$\vec{v} = \vec{w} - \vec{u}.$$

[Add "head-to-tail" & subtract "tail-to-tail".]

- Scalar Multiplication



etc.

Changes the length/orientation but not the direction.

We have the following basic rules of "vector arithmetic":

For all vectors $\vec{u}, \vec{v}, \vec{w}$ (with the same number of coordinates) and for all scalars r, s we have the following obvious rules:

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
- $\vec{u} + \vec{0} = \vec{u}$

[Note: The "zero vector" is

$$\vec{0} = \langle 0, 0, \ldots, 0 \rangle.$$

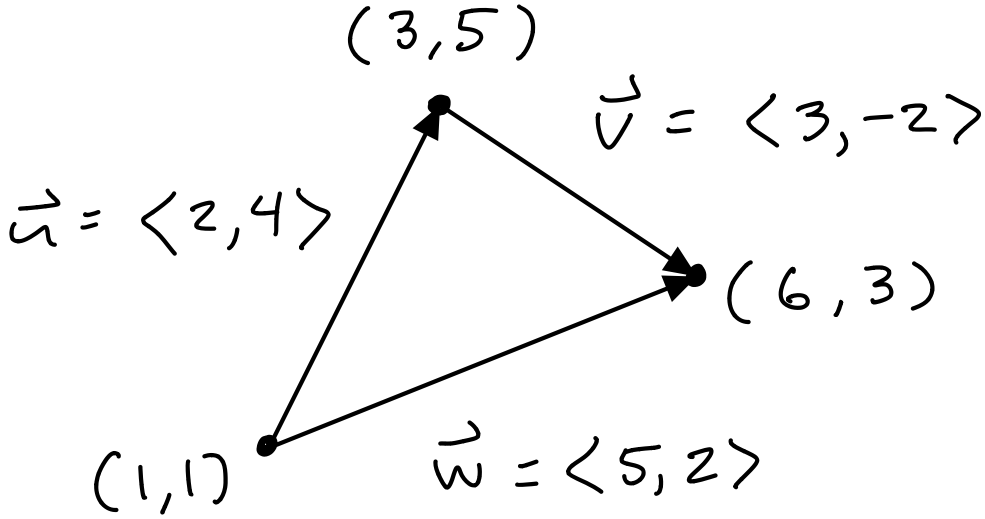
It is special because it doesn't really have a "direction".]

- $\vec{u} + (-\vec{u}) = \vec{0}$
- $r(s\vec{u}) = (rs)\vec{u}$
- $(r+s)\vec{u} = r\vec{u} + s\vec{u}$
- $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
- $1\vec{u} = \vec{u}$
- $0\vec{u} = \vec{0}$.

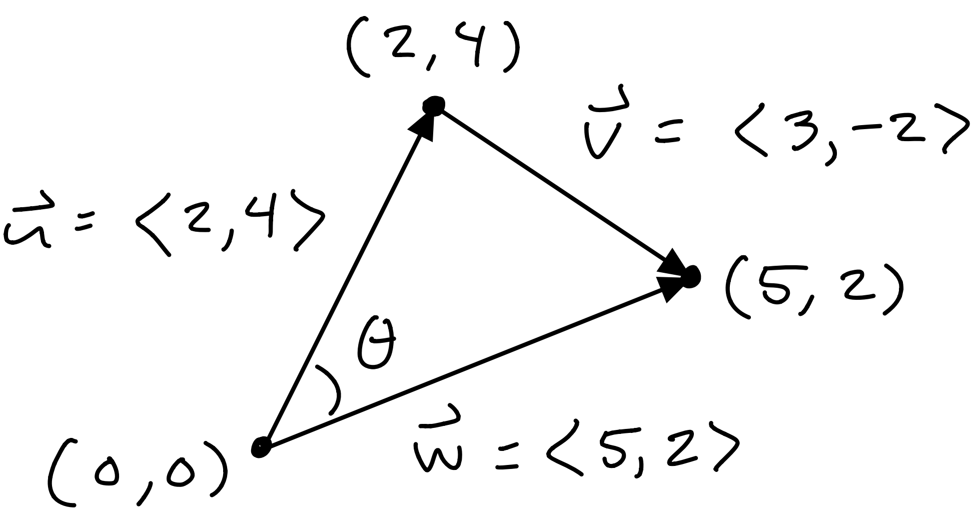
See page 112 in openstax. ///

[See HW 1 Problem 4 ...]

Example: Consider the triangle



Note that we can move the triangle around without changing the components of the vectors:



Let's compute the side lengths

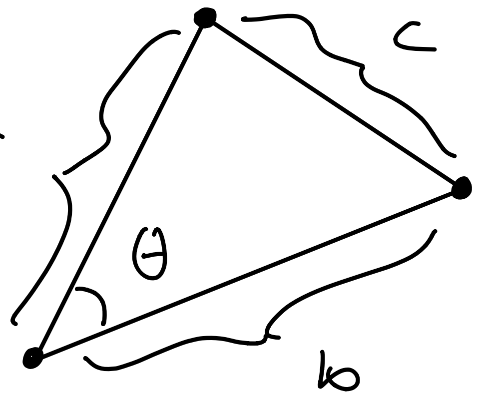
$$\|\vec{u}\| = \|\langle 2,4 \rangle\|$$
$$= +\sqrt{2^2 + 4^2} = +\sqrt{20}$$

$$\|\vec{v}\| = \|\langle 3,-2 \rangle\|$$
$$= +\sqrt{3^2 + (-2)^2} = +\sqrt{13}$$

$$\|\vec{w}\| = \|\langle 5,2 \rangle\|$$
$$= +\sqrt{5^2 + 2^2} = +\sqrt{29}$$

What about the angles?

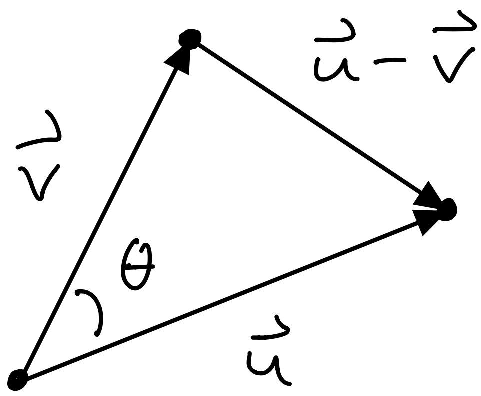
What do you remember from triogonometry? Maybe you have seen the "Law of Cosines" which is a generalization of the Pythagorean Theorem:



$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta}$$

If θ is a right angle then $\cos\theta = 0$ and this becomes the Pythagorean Theorem!

What does this have to do with vectors?



If $a = \|\vec{v}\|$, $b = \|\vec{u}\|$ and $c = \|\vec{u} - \vec{v}\|$ then the Law of cosines tells us that

$$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

On the other hand, we can use the arithmetic of vectors to compute $\|\vec{u} - \vec{v}\|$.

Definition of the Dot Product.

Given two vectors in "n-dim space"

$$\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$$

$$\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$$

we define their <u>dot product</u>

$$\vec{u} \bullet \vec{v} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{\text{this is a scalar}}$$

these are vectors

[Dot product is also called the "scalar product" or the "inner product" of vectors.]

The definition seems a bit strange but we will see that it is <u>extremely useful</u>.

First some more rules of vector arithmetic (pg 147)

- $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
- $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$
- $s(\vec{u} \bullet \vec{v}) = (s\vec{u}) \bullet \vec{v} = \vec{u} \bullet (s\vec{v})$
- $\vec{v} \bullet \vec{v} = \|\vec{v}\|^2$

For this last identity, we observe that

$$\vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + \cdots + v_n v_n$$
$$= v_1^2 + v_2^2 + \cdots + v_n^2$$

and so

$$\|\vec{v}\| = +\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
$$\|\vec{v}\| = +\sqrt{\vec{v} \cdot \vec{v}}$$
$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} \quad \checkmark$$

[In 2D this is the Pyth Thm. In n-dimensions we can just take it as the <u>definition</u> of the length of a vector.]

On HW 1 Problem 4 you will apply these rules to show that

For any vectors $\vec{u}$ & $\vec{v}$ we have

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$\vdots$ some arithmetic

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2(\vec{u}\cdot\vec{v})$$

[Hint: Don't think! Just apply the rules mechanically.]

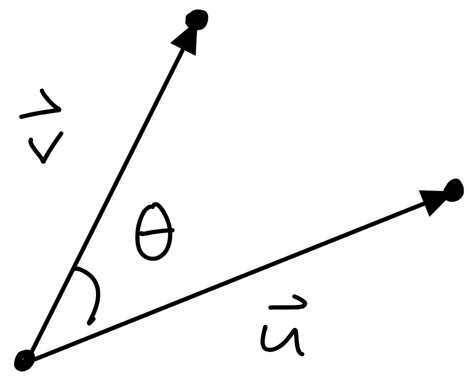
Comparing this to the previow formula

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Gives the following important result.

The Fundamental Theorem of the Dot Product.

Place vectors $\vec{u}$ & $\vec{v}$ tail to tail and let θ be the angle between them



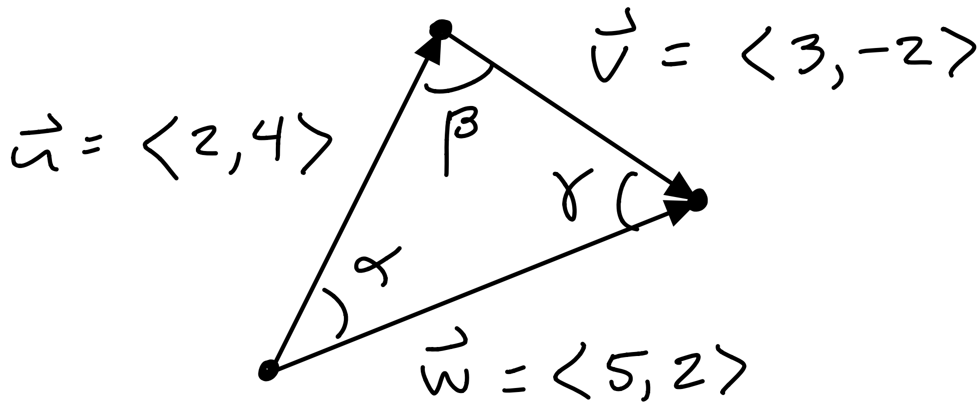
$$\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

This allows us to compute angles using only the dot product:

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\sqrt{\vec{u} \cdot \vec{u}} \sqrt{\vec{v} \cdot \vec{v}}}$$

Example: Let's use this to compute the angles in our favorite triangle.



Recall: $\|\vec{u}\| = \sqrt{20}$

$\|\vec{v}\| = \sqrt{13}$

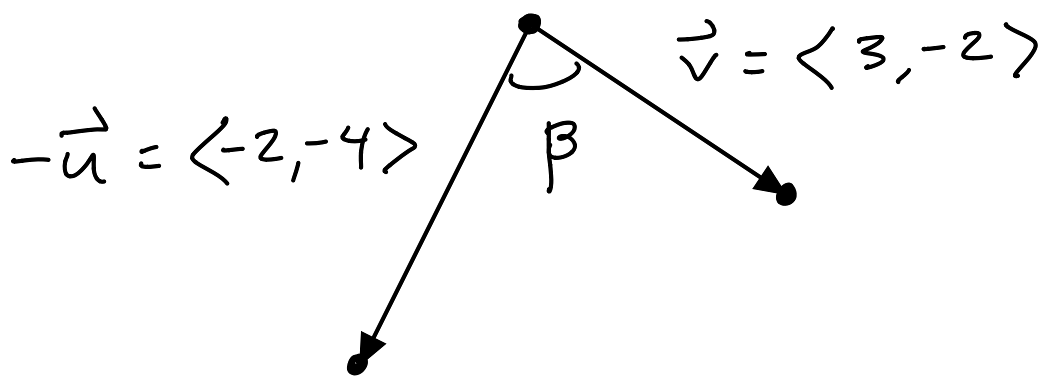
$\|\vec{w}\| = \sqrt{29}$

To compute α:

$$\cos \alpha = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} = \frac{2 \cdot 5 + 4 \cdot 2}{\sqrt{20}\sqrt{29}}$$

$\longrightarrow \quad \alpha \approx 41.68^\circ$

To compute β: There is a problem because $\vec{u}$ & $\vec{v}$ are not tail-to-tail. Easy fix: Just use $-\vec{u}$ instead

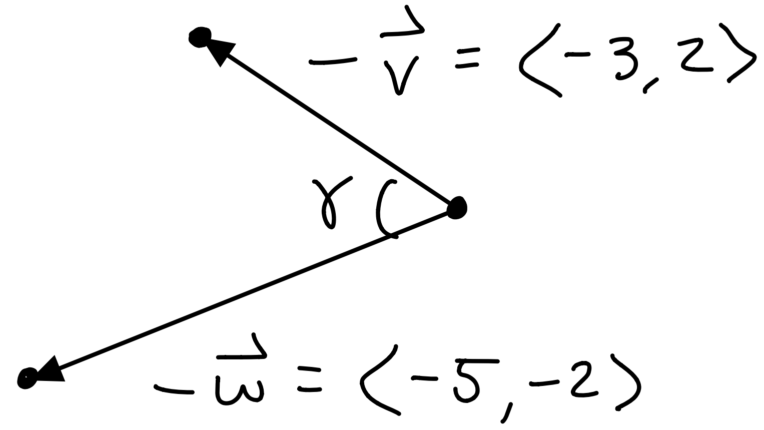


Then the formula says

$$\cos\beta = \frac{(-\vec{u}) \cdot \vec{v}}{\|-\vec{u}\| \, \|\vec{v}\|} = \frac{(-2)(3) + (-4)(-2)}{\sqrt{20}\sqrt{13}}$$

$\longrightarrow \quad \beta \approx 82.85^\circ$

Finally we compute γ by using $-\vec{v}$ & $-\vec{w}$:



$$\cos \gamma = \frac{(-\vec{v}) \cdot (-\vec{w})}{\|-v\| \, \|-\vec{w}\|} = \frac{(-3)(-5) + 2(-2)}{\sqrt{13}\,\sqrt{29}}$$

$$\rightsquigarrow \gamma \approx 55.49^\circ$$

Remark :

$$\alpha + \beta + \gamma \approx 41.63^\circ + 88.85^\circ + 55.49^\circ$$
$$\approx 180^\circ \quad \checkmark$$

[See HW 1 Problem 3.]