HW 1 will be due Friday <u>before</u> the lecture (upload pdf to Bb).

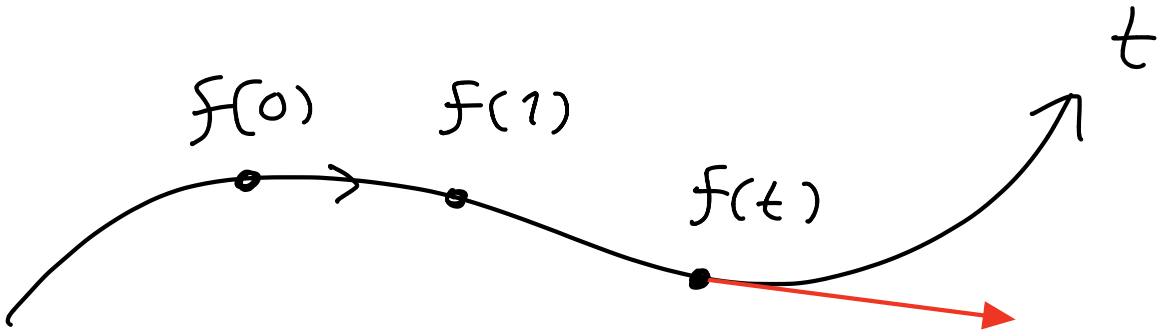
Quiz 1 will be during the first 20 minutes of Monday's lecture, ~ 11:40 — 12:00.

Last time § 1.1 & 1.2 in openstax.

Review: A function $f : \mathbb{R} \to \mathbb{R}^2$ can be thought of as a "parametrized curve" in the plane $\mathbb{R}^2$.

$$f(t) = (\underbrace{x(t), y(t)}_{\text{the position of the particle at time } t})$$

You should think of this as a picture:

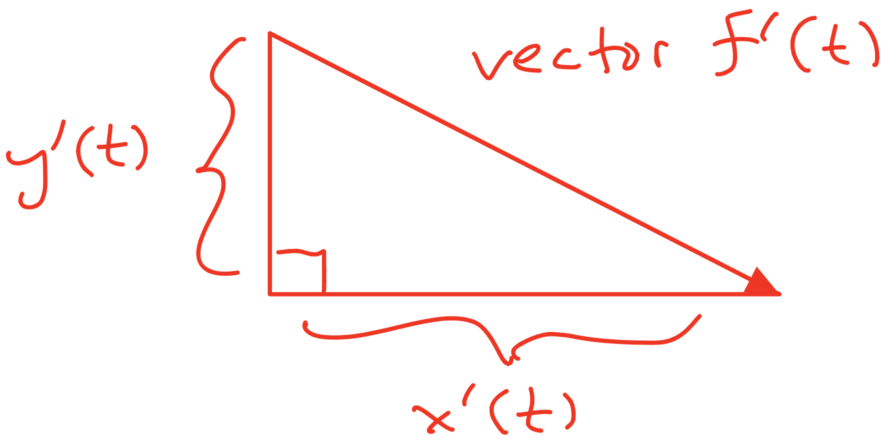


the red arrow is the "instantaneous velocity" at time t

$$f'(t) = (x'(t), y'(t))$$

$$\frac{dF}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

The "instantaneous speed" at time t is the <u>length</u> of the velocity vector, which we compute using the Pythagorean theorem:



$$\text{speed} = \text{length of vector } f'(t)$$
$$= \| f'(t) \|$$
$$= \sqrt{(x'(t))^2 + (y'(t))^2}.$$

The distance traveled (the arc length) by the particle between times $t=a$ & $t=b$ is

$$\text{distance} = \int \text{speed}\, dt$$
$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Arc length integrals are usually very difficult to solve. But here is an example that is not so bad.

Example: The curve defined by

$$y^2 = x^3$$

is called a semi-cubical parabola (also called "Neile's parabola"). It can be parametrized as follows:

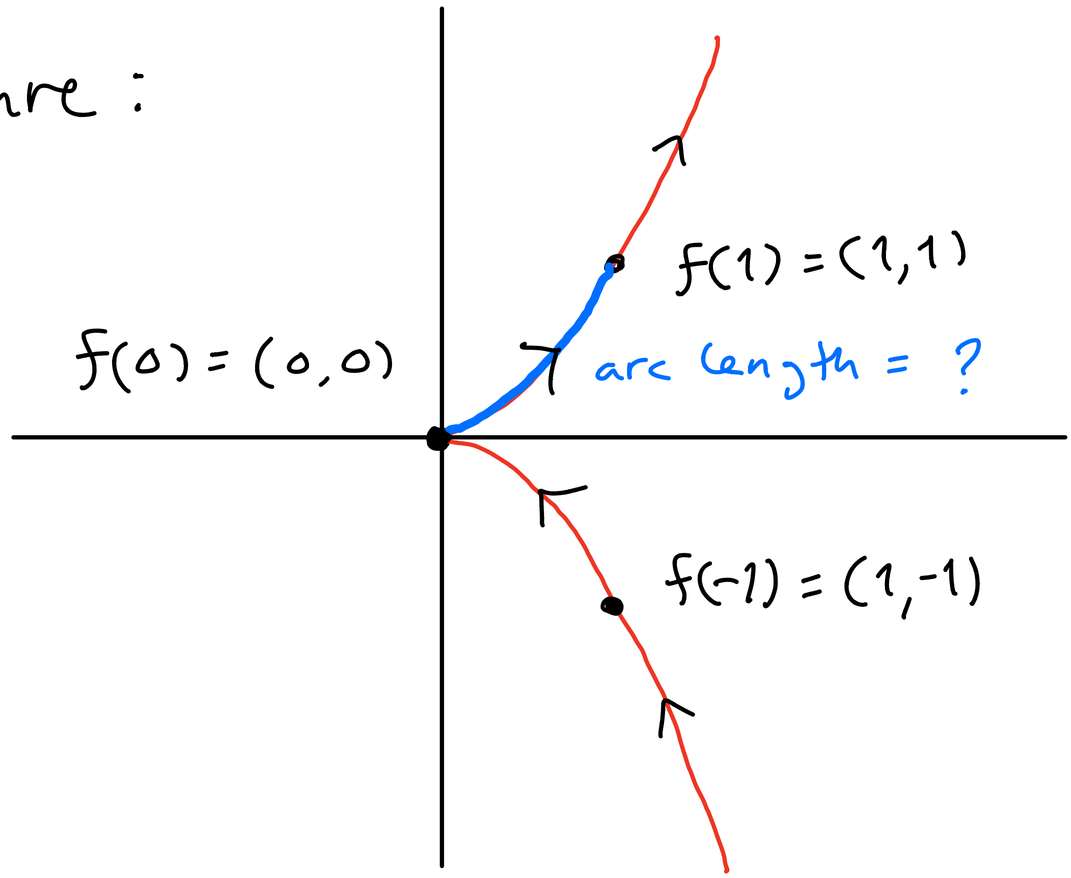
$$f(t) = (x(t), y(t)) = (t^2, t^3).$$

[Check: Indeed, if $x = t^2$ and $y = t^3$ for some t, then

$$y^2 = (t^3)^2 = t^6 = (t^2)^3 = x^3,$$

so the point $f(t) = (x, y) = (t^2, t^3)$ is on the curve $y^2 = x^3$.]

Picture:



The velocity at time t is

$$f'(t) = (x'(t), y'(t))$$
$$= \left(\frac{d}{dt} t^2, \frac{d}{dt} t^3\right)$$
$$= (2t, 3t^2).$$

The speed at time t is

$$\| f'(t) \| = \sqrt{(2t)^2 + (3t^2)^2}$$

$$= \sqrt{4t^2 + 9t^4}$$

$$= \sqrt{t^2(4+9t^2)}$$

$$= |t|\sqrt{4+9t^2}$$

Note: As t goes from $-\infty$ to 0 the particle slows down and then stops. Then it changes direction and speeds up as t goes from 0 to $+\infty$.

The arc length of this curve was first computed by William Neile in 1657, before Calculus was invented! But we will use Calculus.

Between times $t=0$ & $t=1$

(so $|t|=t$) the arc length is

$$\text{arc length} = \int_0^1 \text{speed}\,dt$$

$$= \int_0^1 t\sqrt{4+9t^2}\,dt$$

Substitute $u = 4+9t^2$

$$du = 18t\,dt$$

$$t\,dt = du/18$$

$$\text{arc length} = \int_{t=0}^{t=1} \sqrt{u}\;t\,dt$$

$$= \int_{u=4}^{u=13} \frac{1}{18} u^{1/2}\,du$$

$$= \frac{1}{18} \frac{u^{3/2}}{3/2} \Bigg|_{u=4}^{u=13}$$

$$\left[\int u^n \, du = \frac{u^{n+1}}{n+1} \right]$$

$$= \frac{1}{27}\left(13^{3/2} - 4^{3/2} \right).$$

EXACTLY !

$$\approx 1.439 \ldots$$

This is a lucky example. Usually an arc length integral is too hard to solve exactly !

Exercise for you: Compute the distance traveled by

$$f(t) = (1+3t, 4-2t)$$

between times $t=0$ & $t=1$.

Solution: velocity vector is

$$f'(t) = \left(\frac{d}{dt}(1+3t), \frac{d}{dt}(4-2t) \right)$$

$$= (3, -2)$$

"constant velocity: always moving in the same direction at the same speed."

Speed is

$$\| f'(t) \| = \sqrt{(3)^2 + (-2)^2}$$

$$= \sqrt{13}$$

Distance traveled between times $t = 0$ & $t = 1$

$$= \int_0^1 \text{speed}\, dt$$

$$= \int_0^1 \sqrt{13}\, dt = \sqrt{13}.$$

Picture: Let $x = 1 + 3t$
$y = 4 - 2t$

Eliminate the t to get

$$\frac{x-1}{3} = t = \frac{y-4}{-2}$$

$$\frac{x-1}{3} = \frac{y-4}{-2}$$

$$-2(x-1) = 3(y-4)$$

$$-2x + 2 = 3y - 12$$

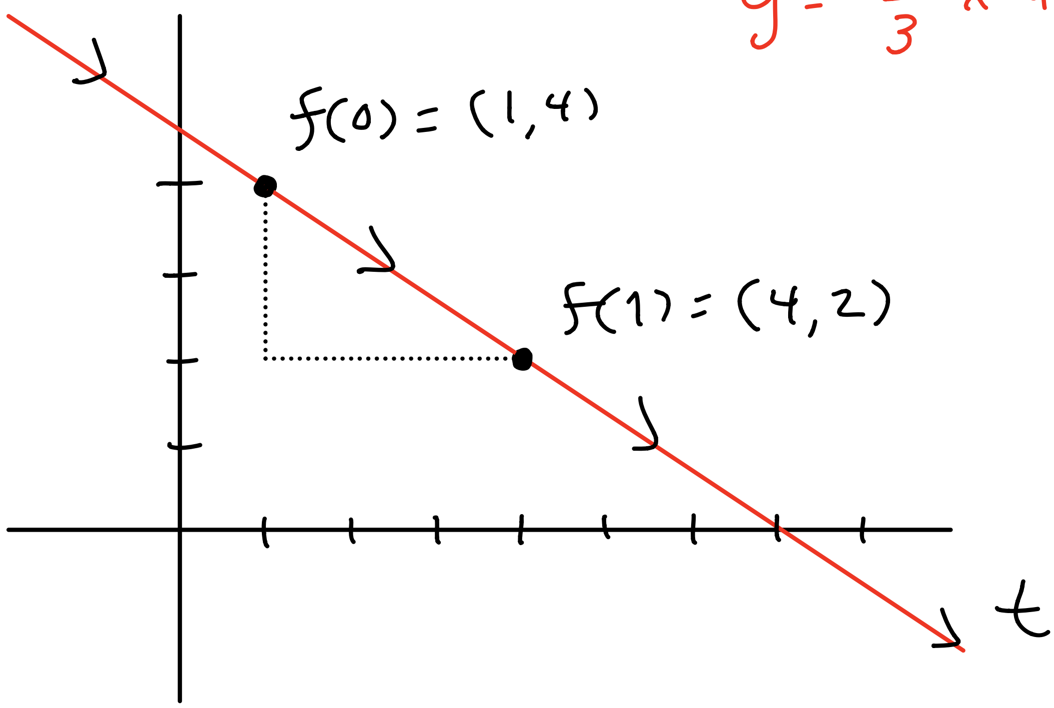
$$3y = -2x + 14$$

$$y = -\frac{2}{3}x + \frac{14}{3}$$

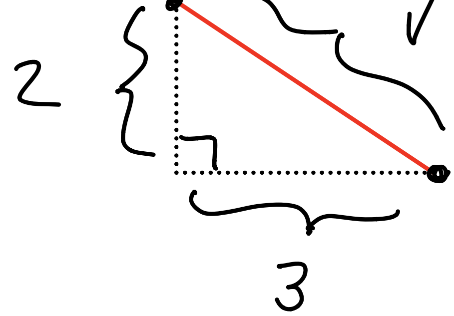
So this curve is a <u>line</u> with slope $-2/3$ and y-intercept $14/3$.

Here is a picture:

$y = -\frac{2}{3}x + \frac{14}{3}$



We could also have computed the distance using just the Pythagorean theorem (no Calculus necessary!):



$\sqrt{3^2+2^2} = \sqrt{13}$ ✓

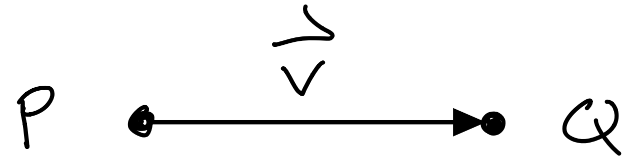
To move all of these ideas up into 3D we need to discuss the language of vectors.

Chapter 2.

What is a "vector"?

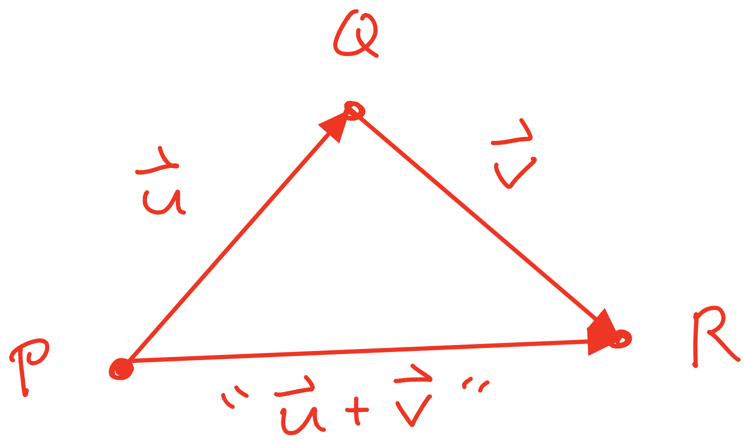
- A quantity with both magnitude & direction (?)
- In this class, a vector is a directed line segment in the plane $\mathbb{R}^2$ or in 3D space $\mathbb{R}^3$.

A vector is determined by two points, a tail & a head:



Notation: $\vec{v} = \overrightarrow{PQ}$ is the vector with tail at point P and head at point Q.

Vectors can be added head-to-tail:



In other words, if $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{QR}$ then

$$\vec{u} + \vec{v} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}.$$

Why do we call this "addition"?

Definition of Coordinates / Components of a vector.

If $P = (x_1, y_1)$ & $Q = (x_2, y_2)$ then we will write

$$\vec{v} = \overrightarrow{PQ} = \langle x_2 - x_1 , y_2 - y_1 \rangle$$

these numbers $x_2 - x_1$, $y_2 - y_1$ are the coordinates/components of the vector $\overrightarrow{PQ}$.

Then using this language, addition of components becomes addition of vectors: Let

$P = (x_1, y_1)$, $Q = (x_2, y_2)$, $R = (x_3, y_3)$

so $\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$

$$\overrightarrow{QR} = \langle x_3 - x_2 , y_3 - y_2 \rangle$$

Then adding the components gives

$$\overrightarrow{PQ} + \overrightarrow{QR}$$

$$= \langle (\cancel{x_2} - x_1) + (x_3 - \cancel{x_2}), (\cancel{y_2} - y_1) + (y_3 - \cancel{y_2}) \rangle$$

$$= \langle x_3 - x_1 , y_3 - y_1 \rangle = \overrightarrow{PR} \checkmark$$

Example: $P = (1,1)$

$Q = (3,5)$

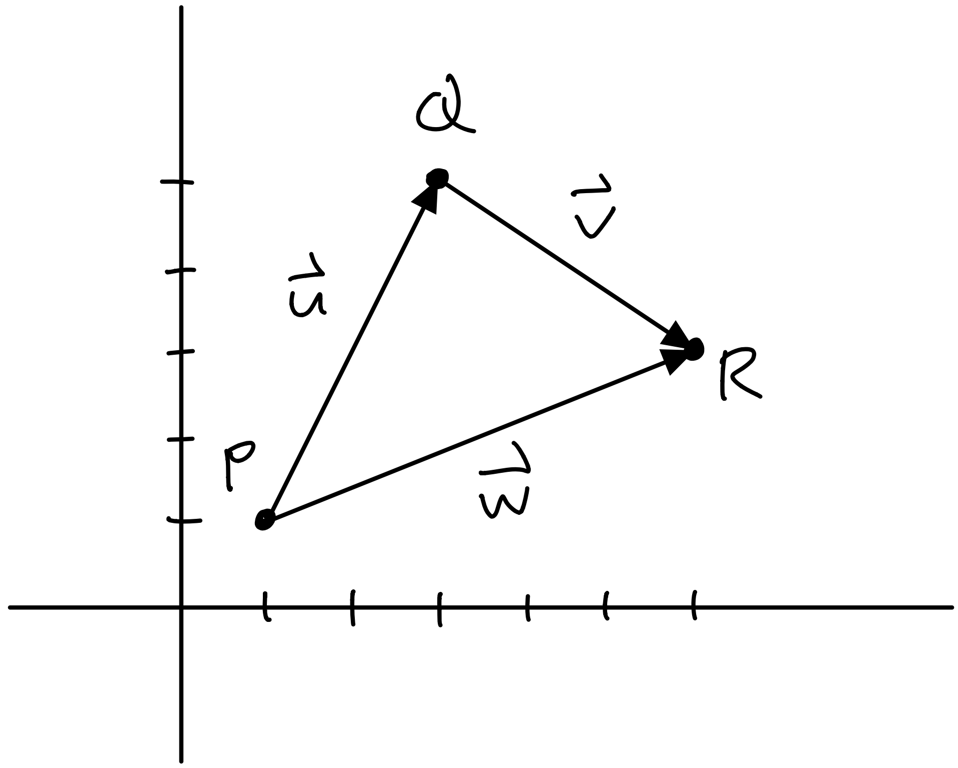
$R = (6,3)$

Let $\vec{u} = \overrightarrow{PQ} = \langle 3-1, 5-1 \rangle = \langle 2,4 \rangle$

$\vec{v} = \overrightarrow{QR} = \langle 6-3, 3-5 \rangle = \langle 3,-2 \rangle$

$\vec{w} = \overrightarrow{PR} = \langle 6-1, 3-1 \rangle = \langle 5,2 \rangle$

These points and vectors form a triangle in the plane:



We verify that $\vec{u} + \vec{v} = \vec{w}$:

$$\begin{aligned} \vec{u} + \vec{v} &= \langle 2, 4 \rangle + \langle 3, -2 \rangle \\ &= \langle 2+3 \, , \, 4+(-2) \rangle \\ &= \langle 5, 2 \rangle \\ &= \vec{w} \quad \checkmark \end{aligned}$$